A Fast Algorithm for Computing Irreducible Triangulations of Closed Surfaces in $\mathbb{E}^d$ and Its Applications to the TriQuad Problem

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Joint work with

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Suneeta Ramaswami
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UNIVERSIDADE FEDERAL DO PARANÁ

1766 RUTGERS THE STATE UNIVERSITY OF NEW JERSEY

Problem Statement

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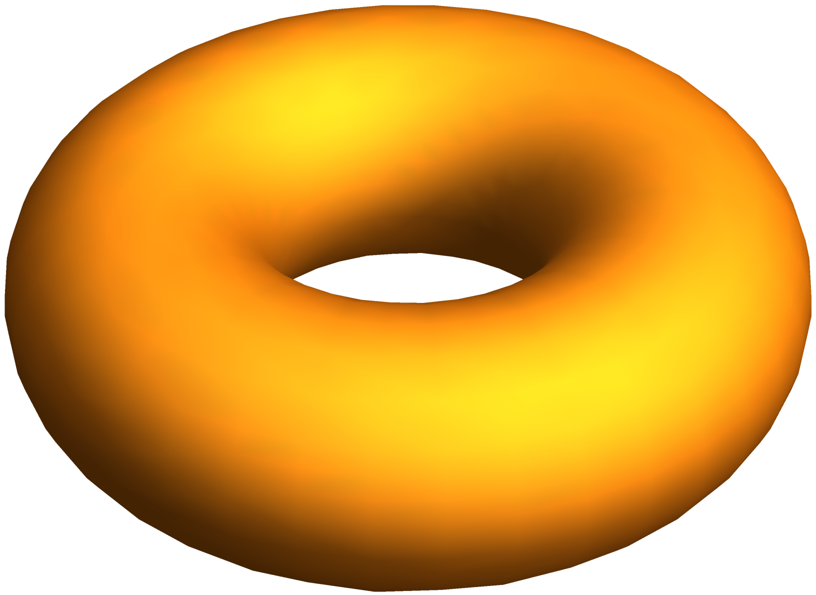
Let $\mathcal{S}$ be a connected, boundaryless, and compact surface in $\mathbb{E}^d$.



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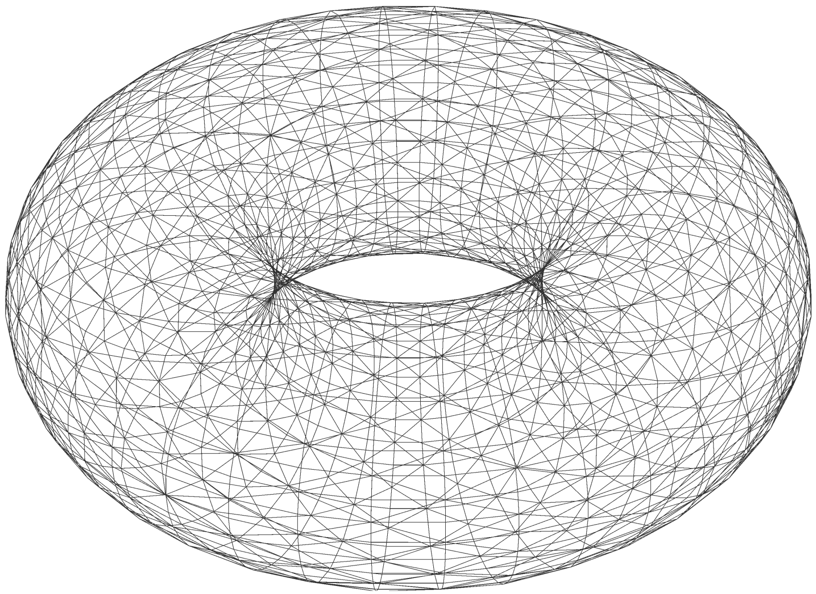
Given a triangulation $\mathcal{T}$ of $\mathcal{S}$,



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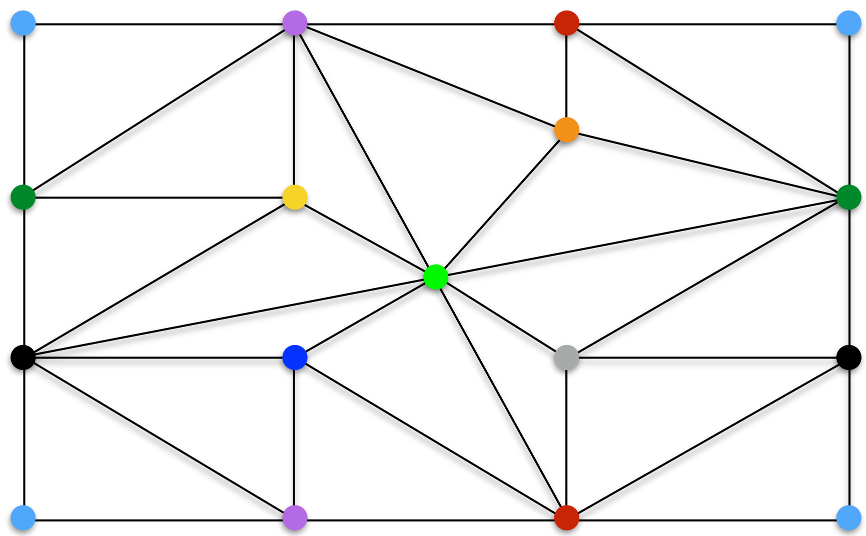
find an **irreducible triangulation** $\mathcal{T}'$ of $\mathcal{S}$ from $\mathcal{T}$:



$$V - E + F = 10 - 30 + 20 = 0 = 2 \cdot 0 = 2 \cdot (1 - 1) = 2 \cdot (1 - g)$$

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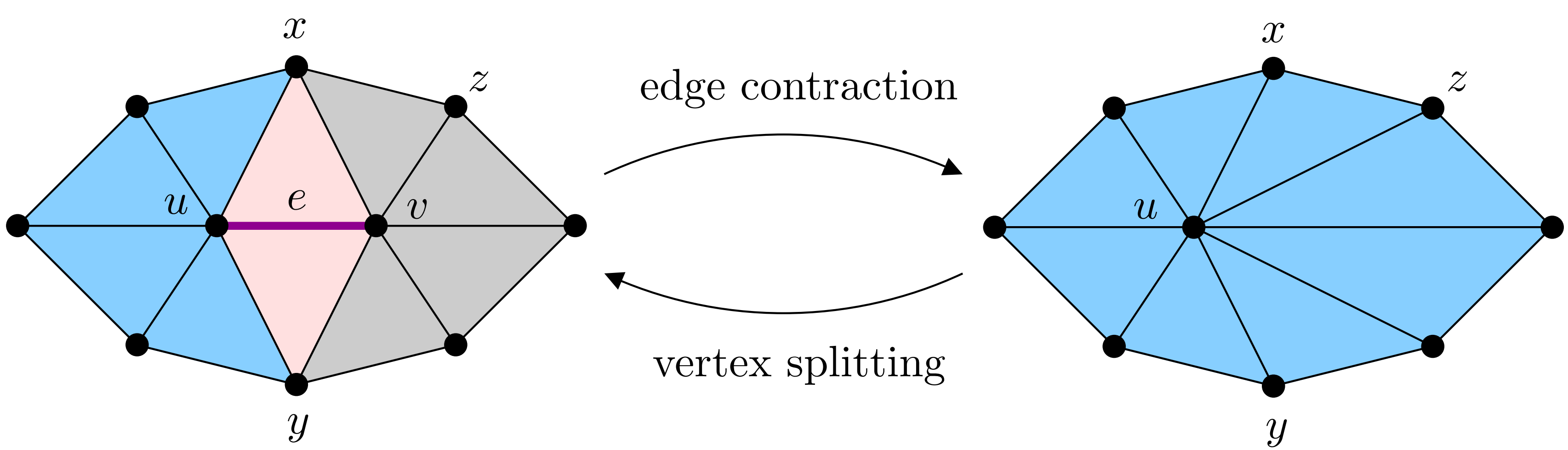
$\mathcal{T}'$ is irreducible if and only if $\mathcal{T}'$ has no contractible edges



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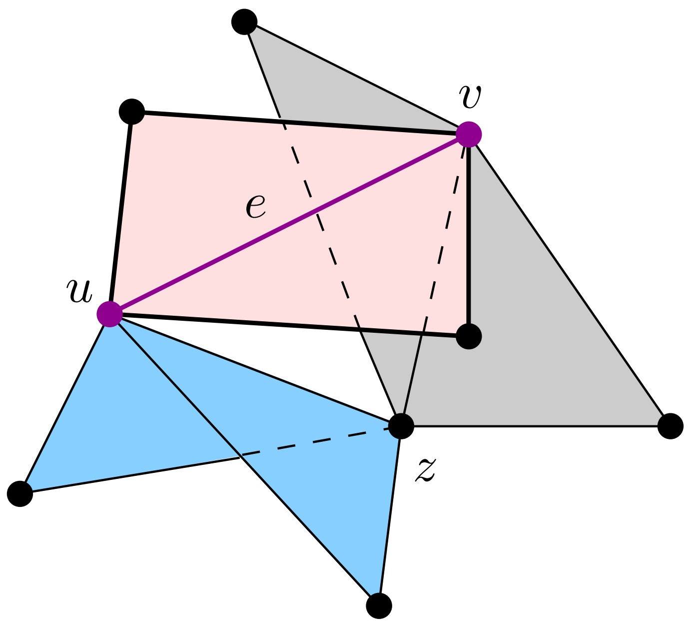

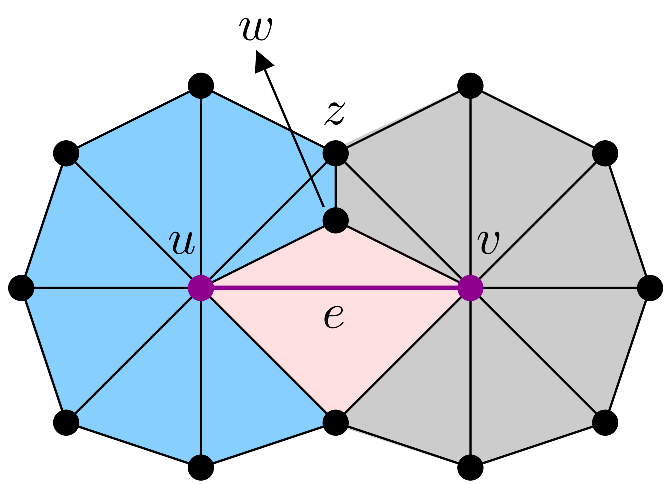

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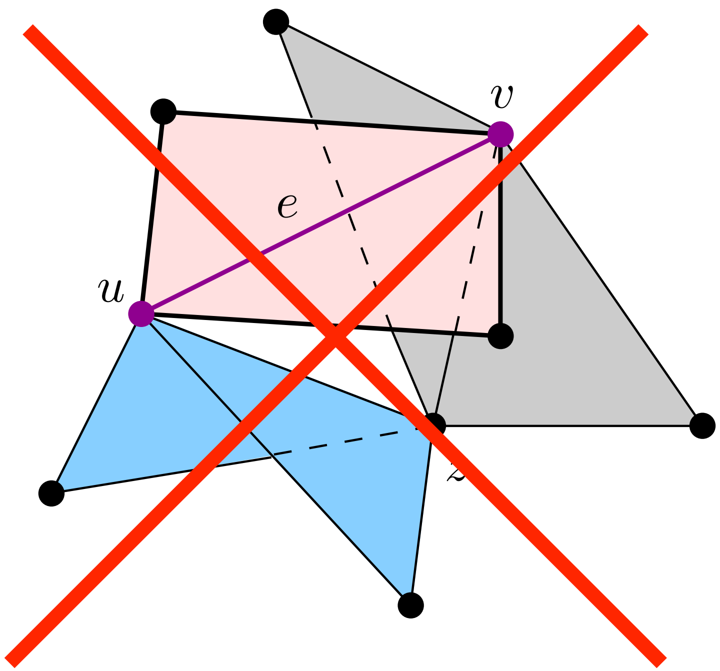

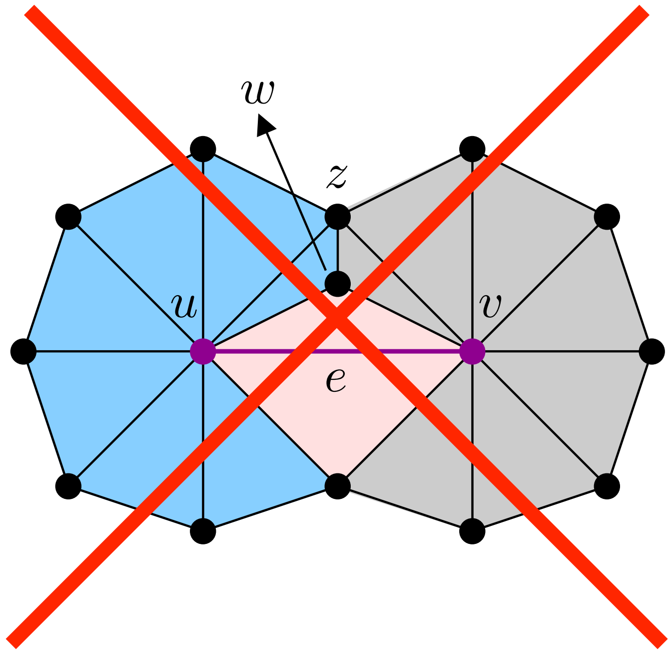

Facts

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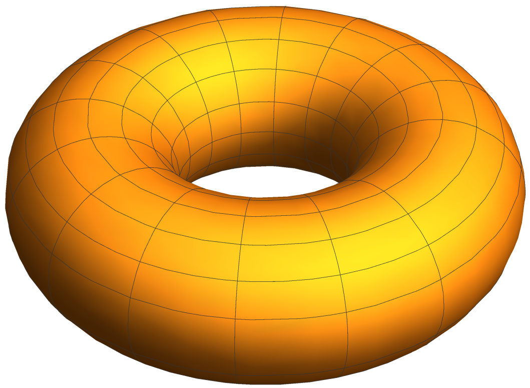
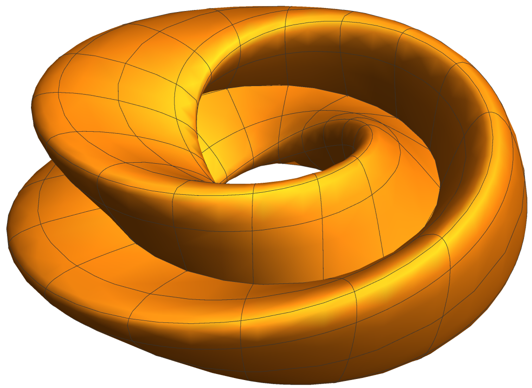
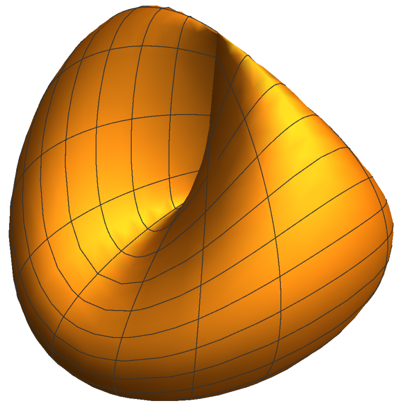
All closed surfaces have finitely many irreducible triangulations (Barnette & Edelson, 1989).



21, with $n_v \in \{7, 8, 9, 10\}$



29, with $n_v \in \{8, 9, 10, 11\}$



2, with $n_v \in \{6, 7\}$

Facts

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If the genus g of $\mathcal{S}$ is positive, then any irreducible triangulation of *smallest* size (known as *minimal*) has $\Theta(\sqrt{g})$ vertices. Also, for any irreducible triangulation of $\mathcal{S}$,

$$n_v \leq 13 \cdot h - 4\,,$$

where h is the *Euler* genus of $\mathcal{S}$ (if $\mathcal{S}$ is orientable, then $h = 2g$. Otherwise, $h = g$.)

(Joret & Wood, 2010)

Facts

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The largest known irreducible triangulation of an orientable surface of genus g has

$$n_v = \left\lfloor \frac{17}{2} \cdot g \right\rfloor$$

(Sulanke, 2006)

Why Should One Care?

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All irreducible triangulations of $\mathcal{S}$ form a "basis" for all triangulations of $\mathcal{S}$.

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All irreducible triangulations of $\mathcal{S}$ form a ″basis″ for all triangulations of $\mathcal{S}$.

Have been used for

- proving the existence of geometric realizations;
- studying properties of diagonal flips on surfaces triangulations;
- characterizing the structure of flexible triangulations;
- finding bounds for the number of cliques in graphs on surfaces.

Why Do We Care?

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The TriQuad Problem

Given a triangulation, $\mathcal{T}$, of $\mathcal{S}$, obtain a quadrangulation, $\mathcal{Q}$, of $\mathcal{S}$ whose vertex set is a (not necessarily proper) superset of the vertex set of $\mathcal{T}$.

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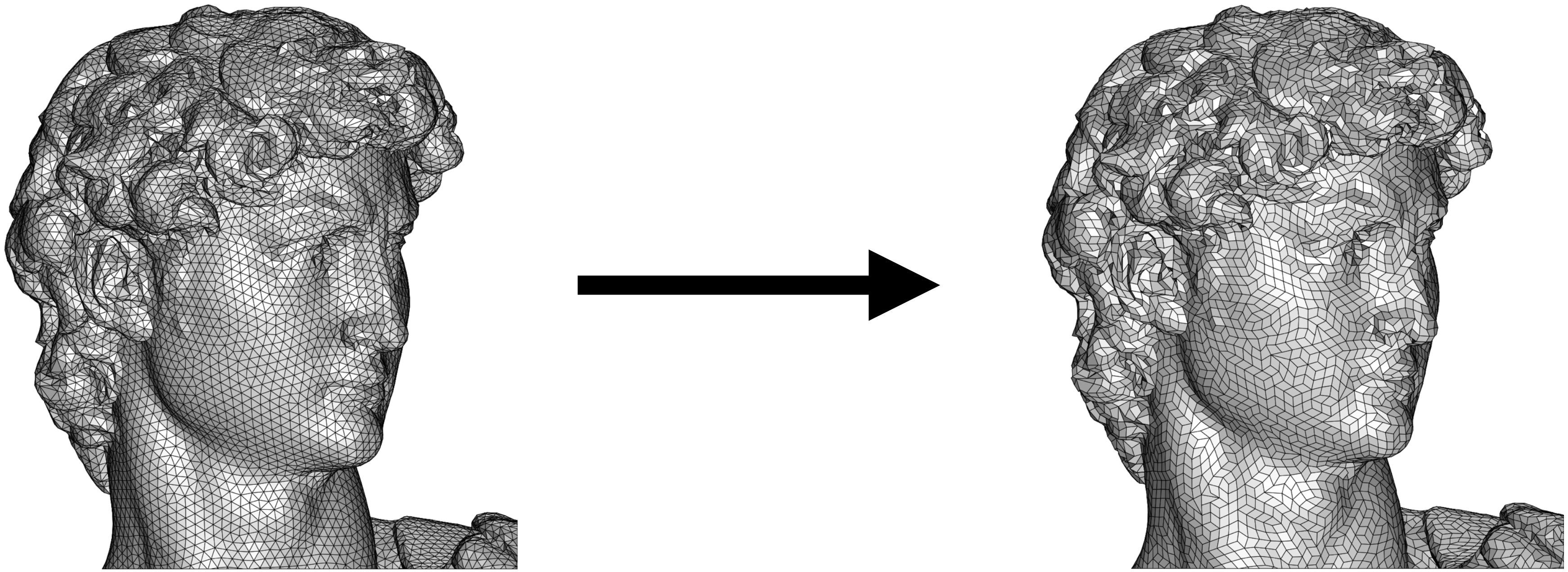

Main Result

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Given a triangulation $\mathcal{T}$ of connected, boundaryless, compact surface $\mathcal{S}$ of genus g, there exists an algorithm for computing an irreducible triangulation $\mathcal{T}'$ of $\mathcal{S}$ in $\mathcal{O}(g^2 + gn_f)$ time if g is positive; otherwise, $\mathcal{T}'$ can be computed in $\mathcal{O}(n_f)$ time, where n_f is the number of triangles of $\mathcal{T}$.

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$\mathcal{O}(n_f \lg n_f + g \ln n_f + g^4)$

Common Strategy

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Let K be the current triangulation (i.e., $K = \mathcal{T}$ initially).

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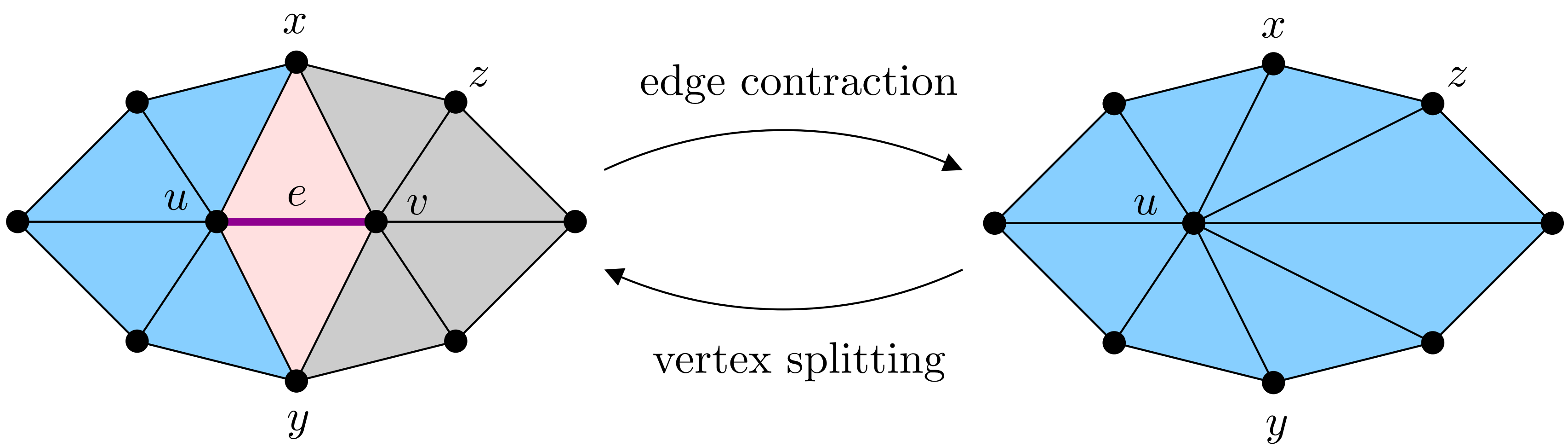
While there exists a *contractible* edge $e = [u, v]$ in the current triangulation K, contract e, identifying v with u and producing triangulation $K - uv$.



Link Condition Test

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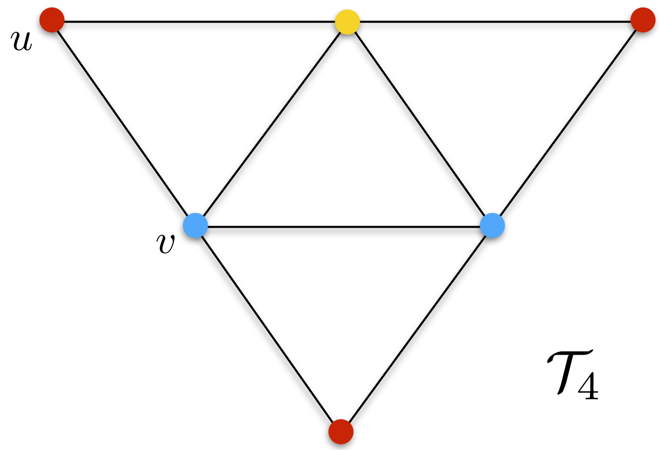
If K is (isomorphic to) $\mathcal{T}_4$ then no edge of K is contractible.



$\mathcal{T}_4$

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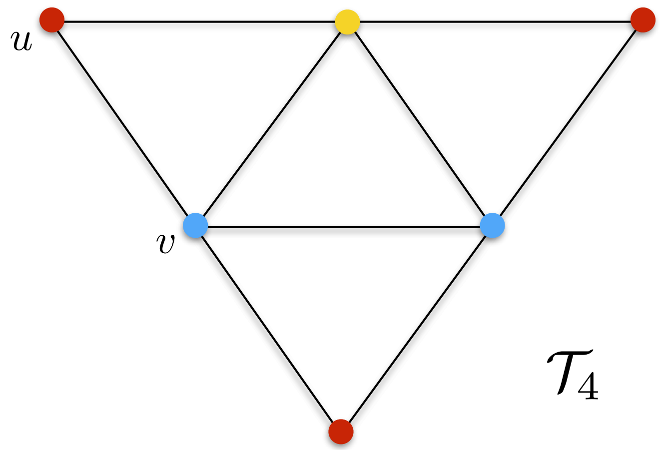

$$K \approx \mathcal{T}_4 \iff d_u = d_v = 3$$

Link Condition Test

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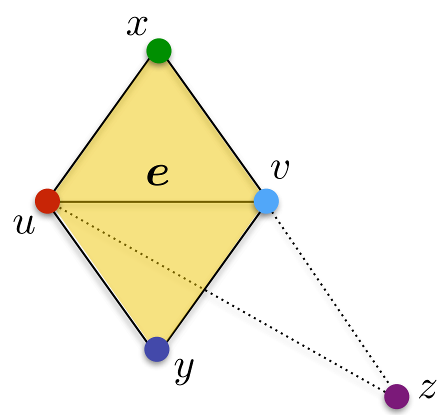
If K is not (isomorphic to) $\mathcal{T}_4$, then $e = [u, v]$ is contractible iff e does not belong to a *critical cycle* (i.e., a 3-cycle that does not bound a triangle) in K.



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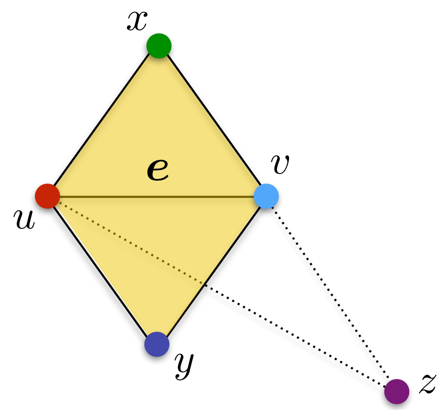

Edge $e = [u, v]$ belongs to a critical cycle iff u and v have a common neighbor other than x and y, where x and y are the vertices of the link of e.

Time Complexity

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(2) The number of link condition tests executed by the algorithm.

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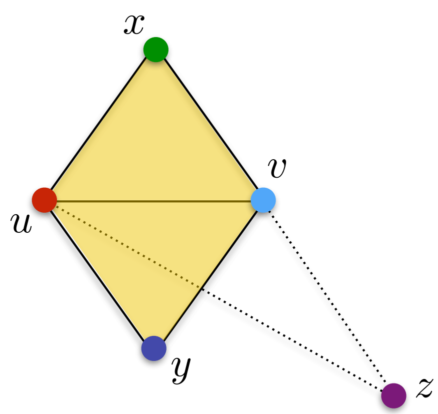

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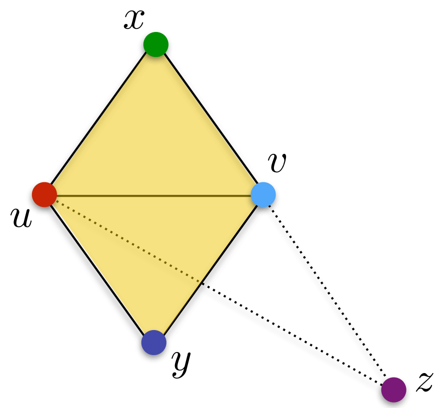

$\Omega(d_u \cdot d_v)$

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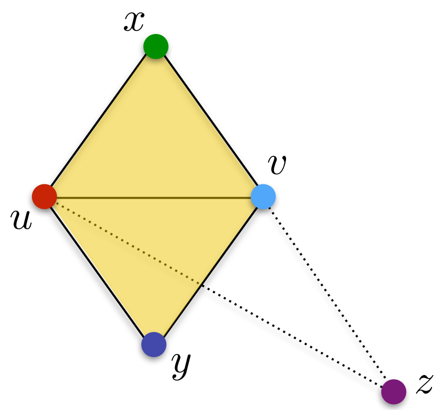

$$\Omega(d_u \cdot d_v)$$

Naïve

$$\Omega(d_u \cdot \lg d_v)$$

Neighborhood dictionary

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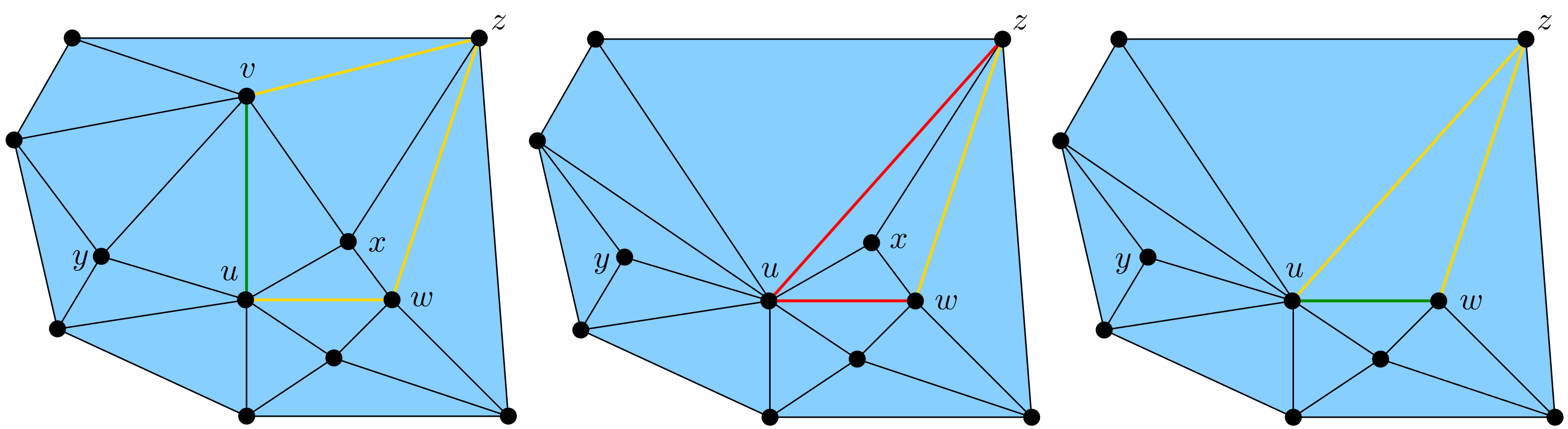

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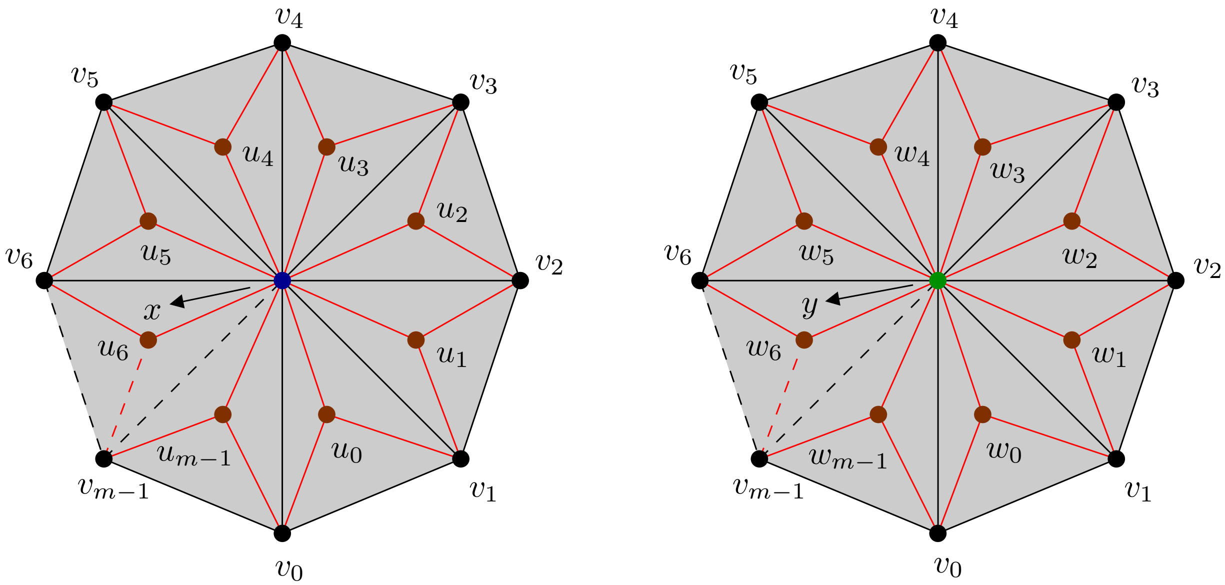

$n_v = 3m + 2$ vertices, with $n_f \in \Theta(n_v)$

Testing all edges $[v_i, x]$ and $[v_i, y]$ *first* may take $\Omega(n_f \lg n_f)$ time

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Contract all edges of the form $[u, v]$ until u becomes *trapped*.

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A vertex w in K is **trapped** if all edges incident on w in K are non-contractible edges. Otherwise, vertex w is said to be **loose**. So, K is an irreducible triangulation if and only if every vertex in K is a trapped one.

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Lemma 1 (Schipper, 1991)
Once u becomes trapped in K, it remains trapped.

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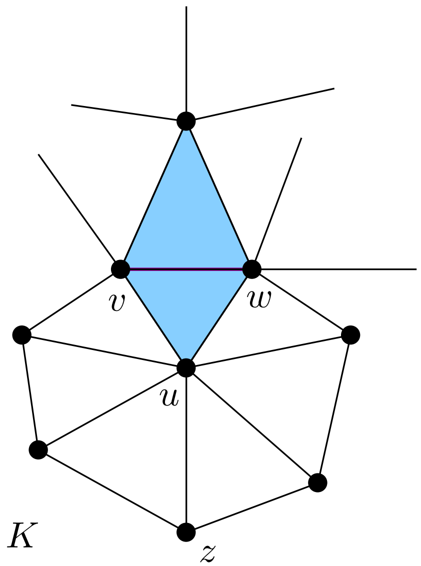

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Pick another vertex u from the current triangulation, K.

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The cost for testing *all* edges *during the processing of* u is

$$\Theta(d_u) + \sum_{v \in \mathcal{A}_u} \Theta(d_v)\,,$$

where $\mathcal{A}_u$ is the set of vertices that are or become adjacent to u.

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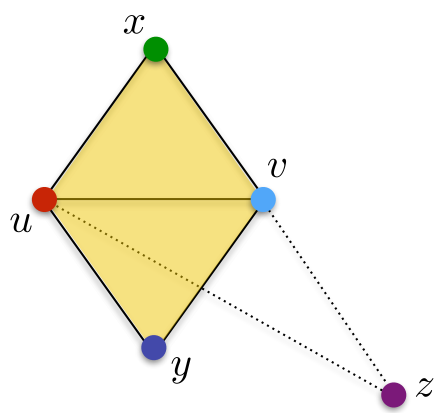

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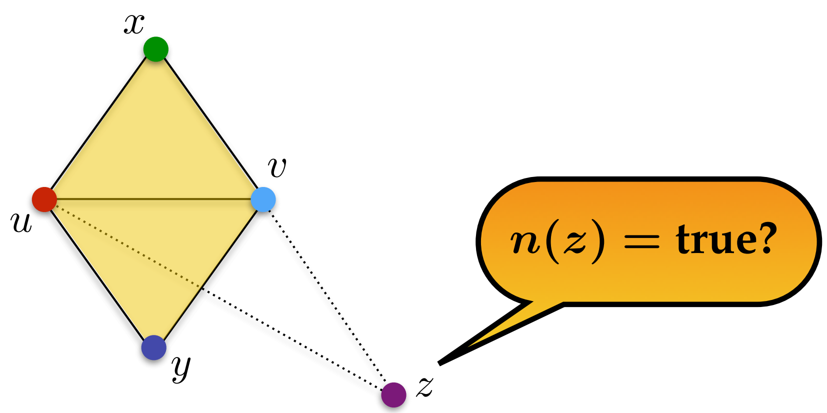

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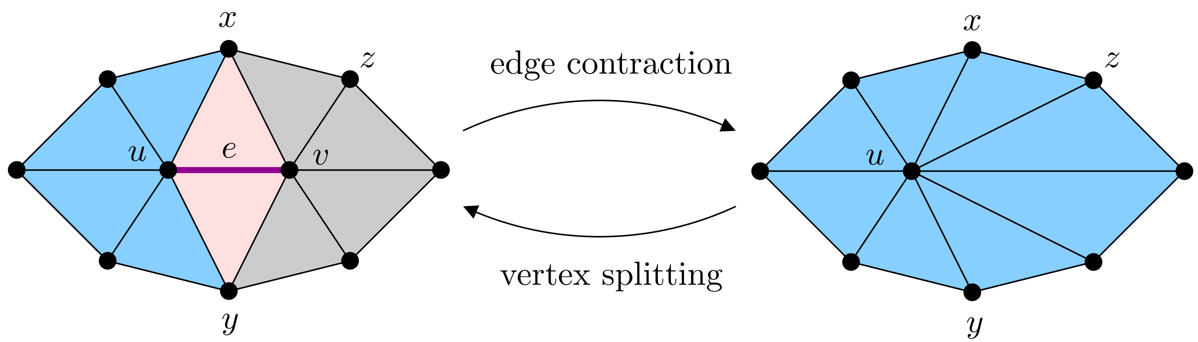

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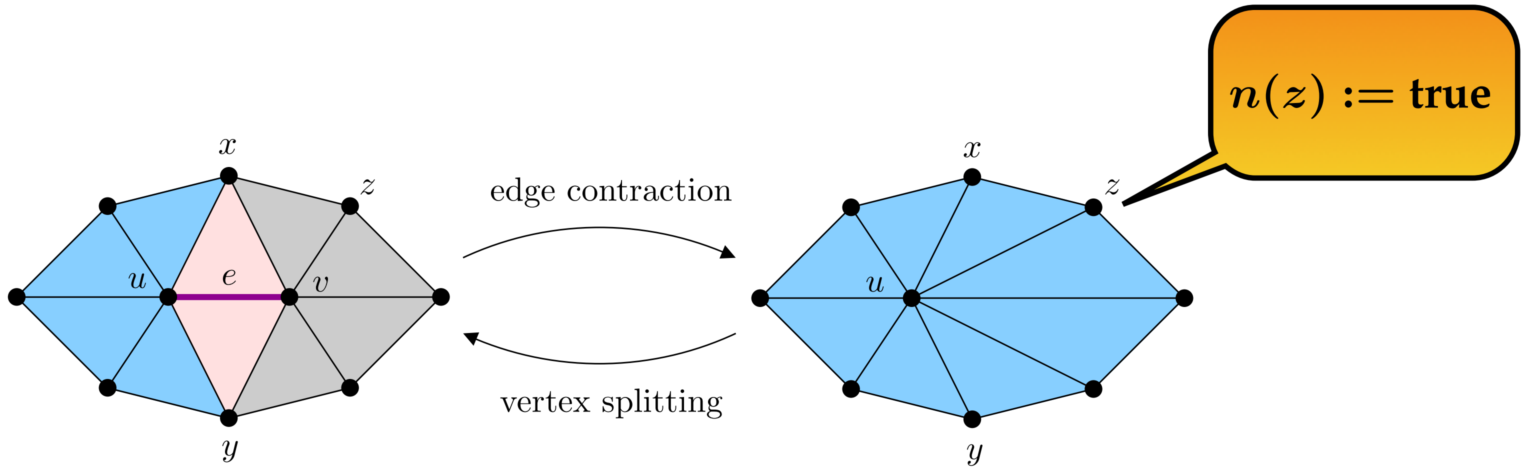

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During the processing of u, edge $[u, v]$ in K is tested against the link condition exactly once if v has not been selected by the algorithm prior to u; else it is not tested.

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Keep a counter for the number of critical cycles every edge $[u, v]$ belongs to.

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Lemma 2
Let K be a surface triangulation, and let v be any vertex of degree 3 in K. If K is (isomorphic to) $\mathcal{T}_4$, then no edge of K is a contractible edge. Otherwise, each of the 3 edges of K incident on v is a contractible edge in K.

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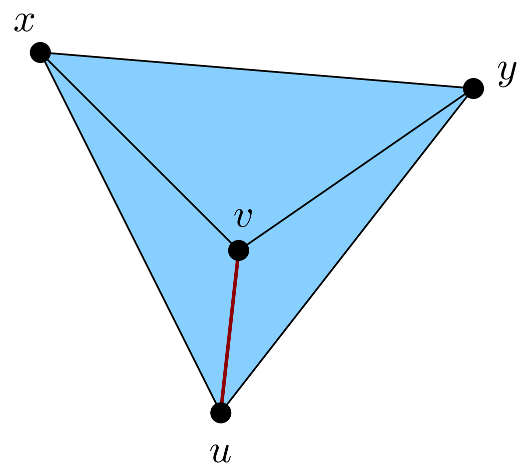

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Lemma 3
Let K be a surface triangulation, and let f be a contractible edge of K. If a non-contractible edge e of K becomes contractible in $K - f$, then f must be incident on a degree-3 vertex v of K and e must belong to $\mathrm{lk}(v, K)$. Moreover, e belongs to a single critical cycle in K, which consists of the edges in $\mathrm{lk}(v, K)$, and this cycle becomes non-critical in $K - f$.

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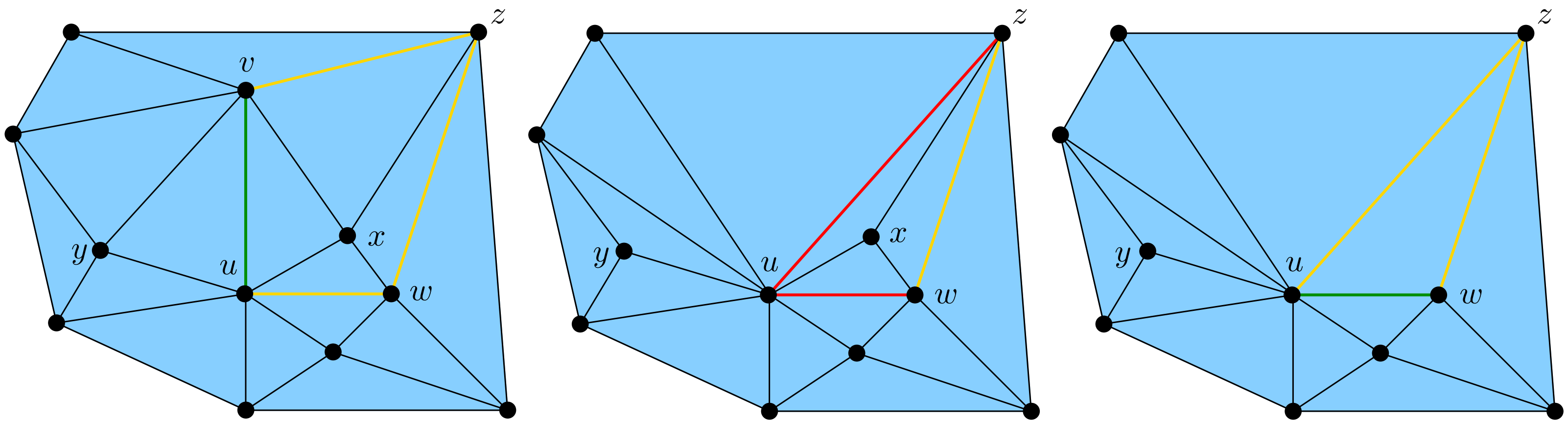

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Lemma 4
Let K be a surface triangulation, and let f be a contractible edge of K. If a contractible edge e of K becomes non-contractible in $K - f$, then the vertex v of f identified with the other vertex, u, of f in $K - f$ is such that

$$d_v > 3\,.$$

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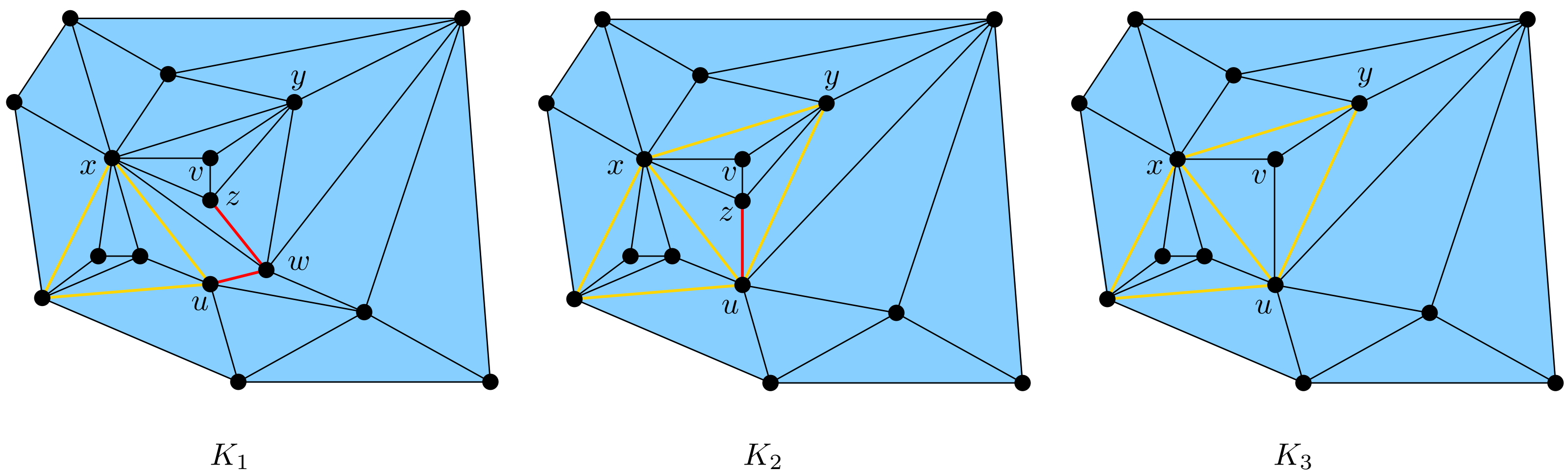

Experimental Results

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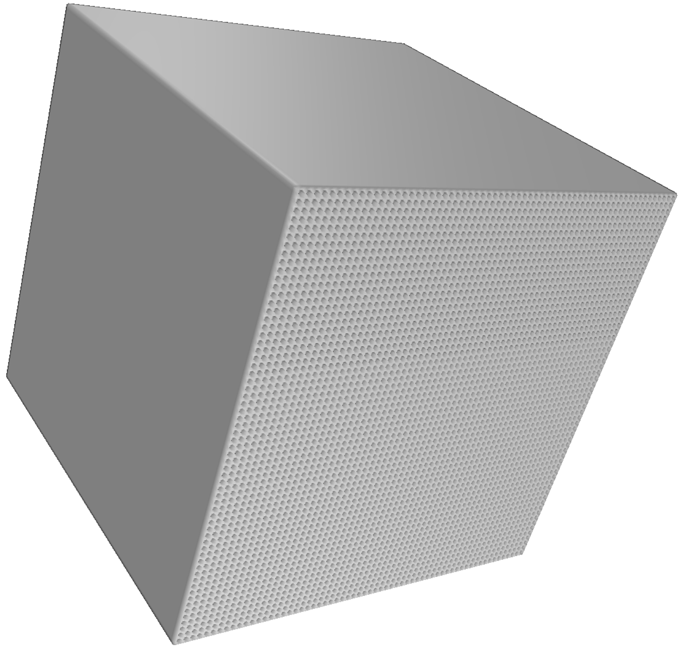
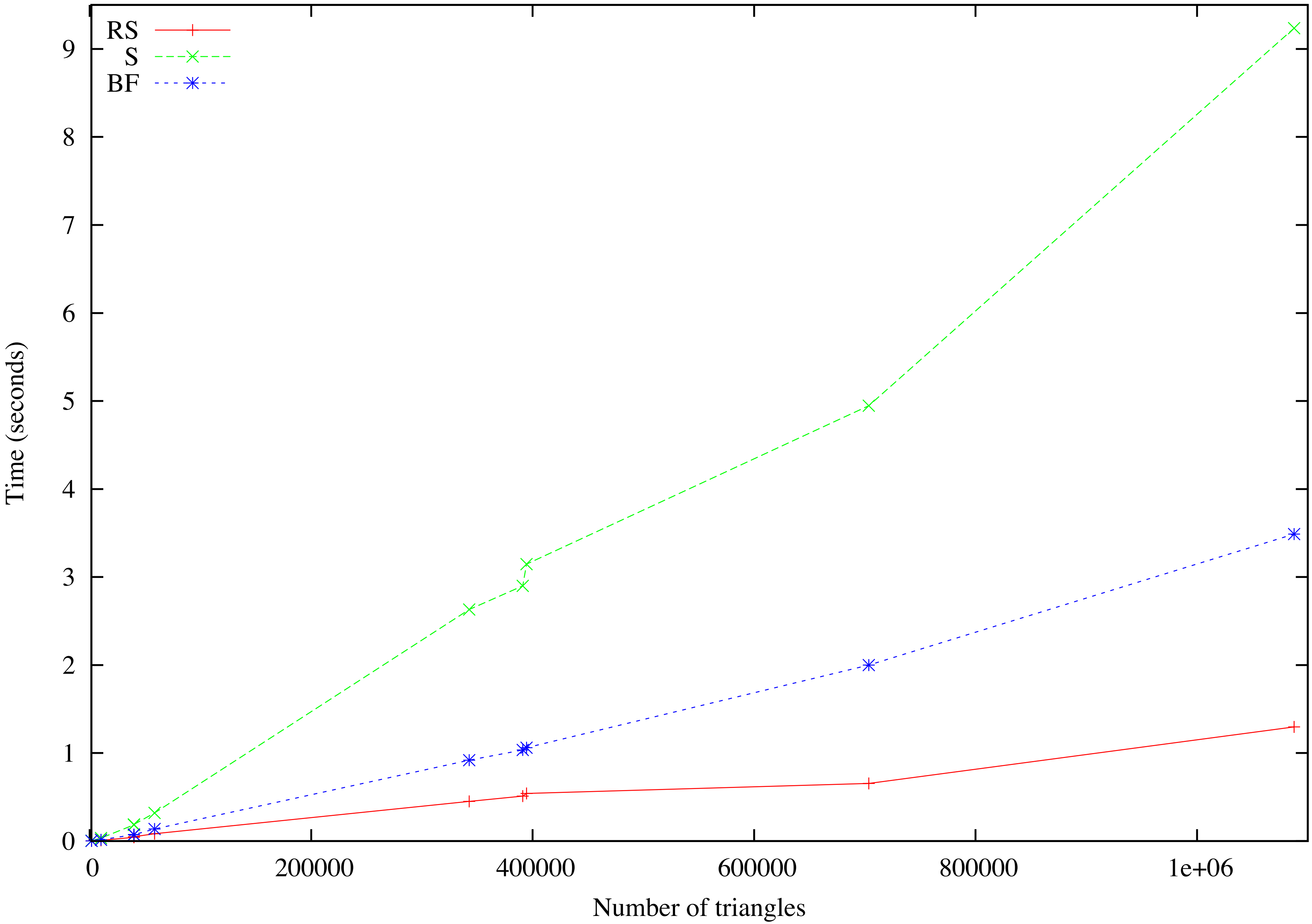
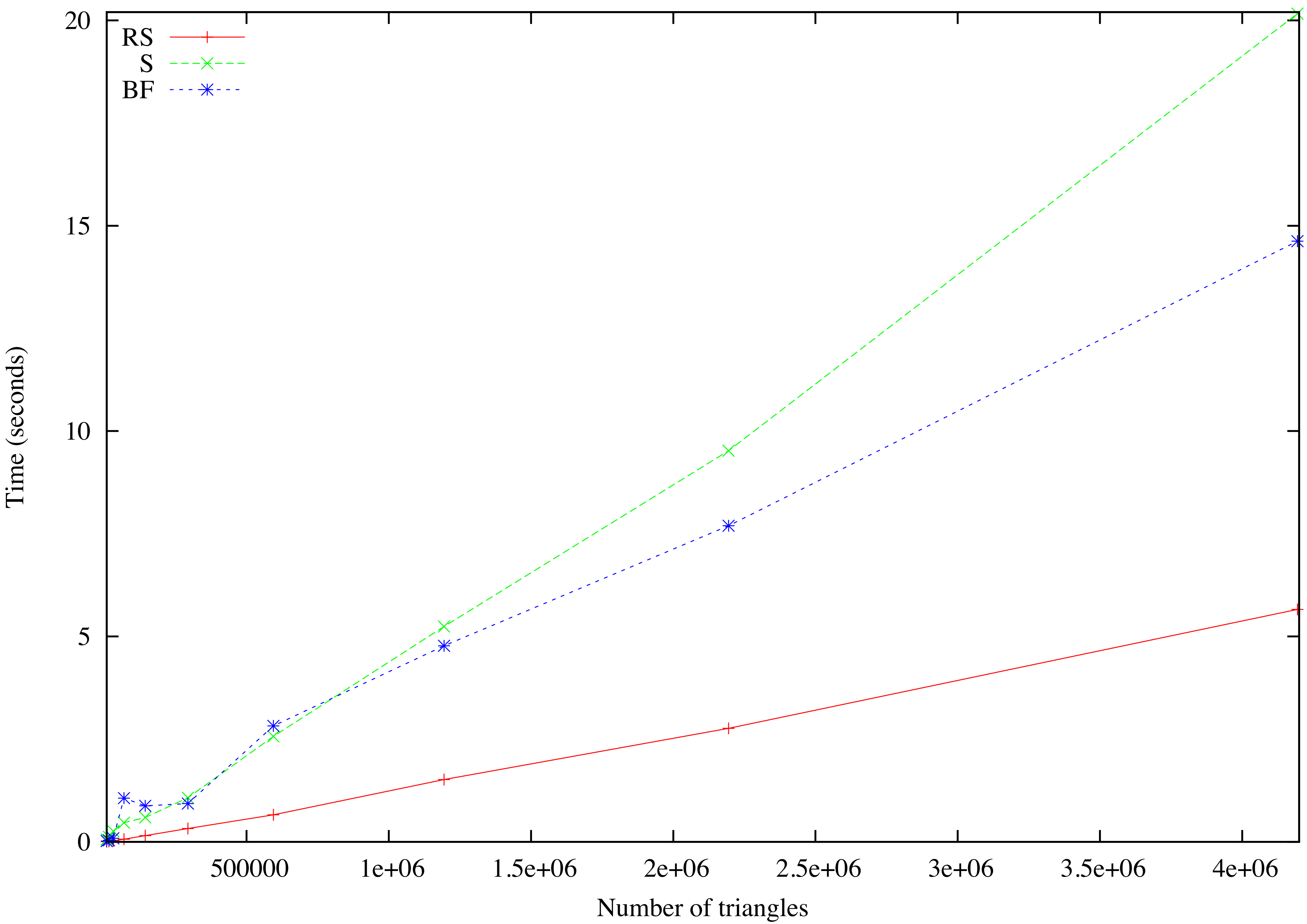
$g \ll n_f$



$g \approx 2 \cdot \sqrt{n_f}$



RS
S
BF
Time (seconds)
0
1
2
3
4
5
6
7
8
9
0
200000
400000
600000
800000
1e+06
Number of triangles


RS
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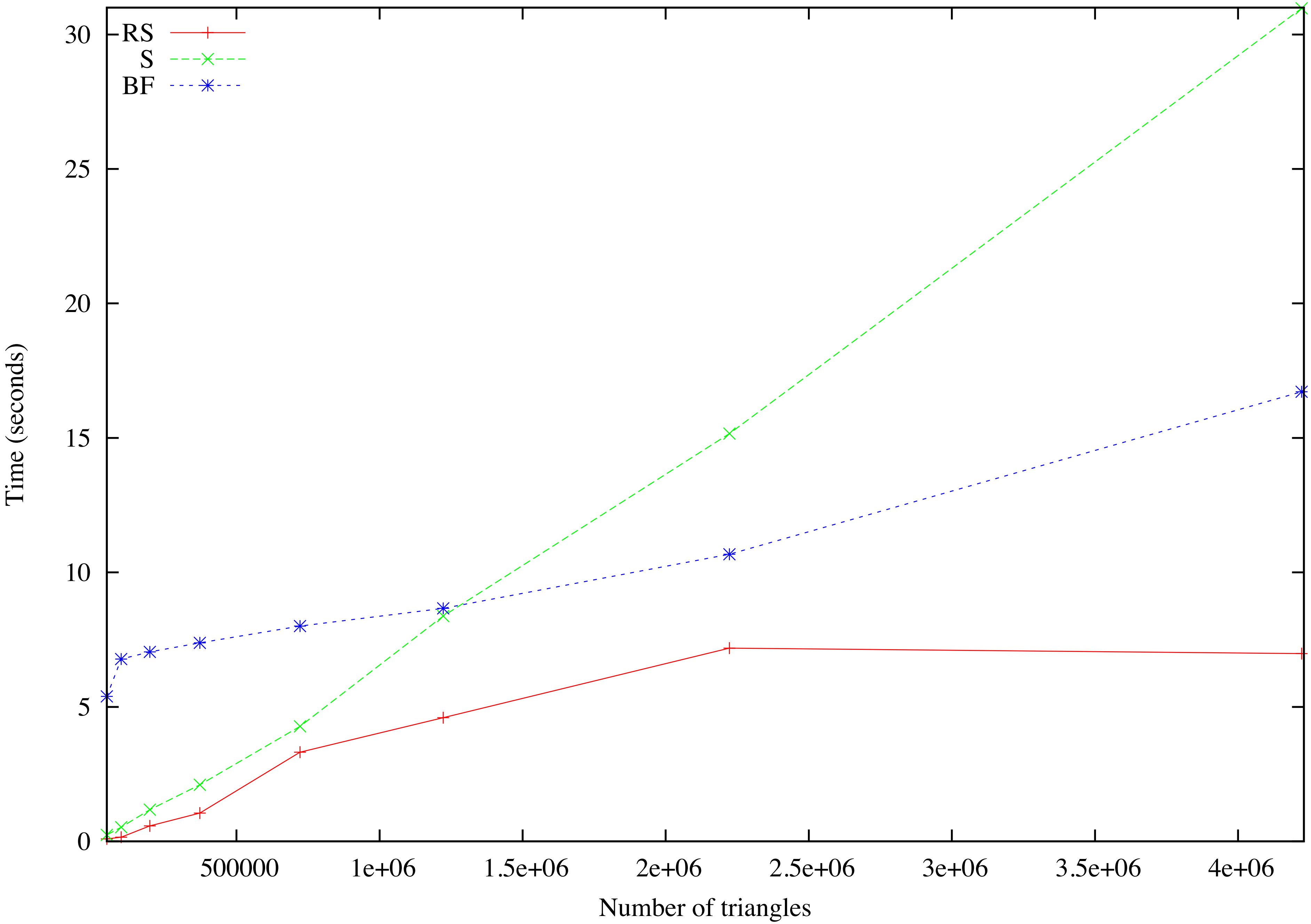
2.5e+06
3e+06
3.5e+06
4e+06
Number of triangles



RS
S
BF
Time (seconds)
Number of triangles
0
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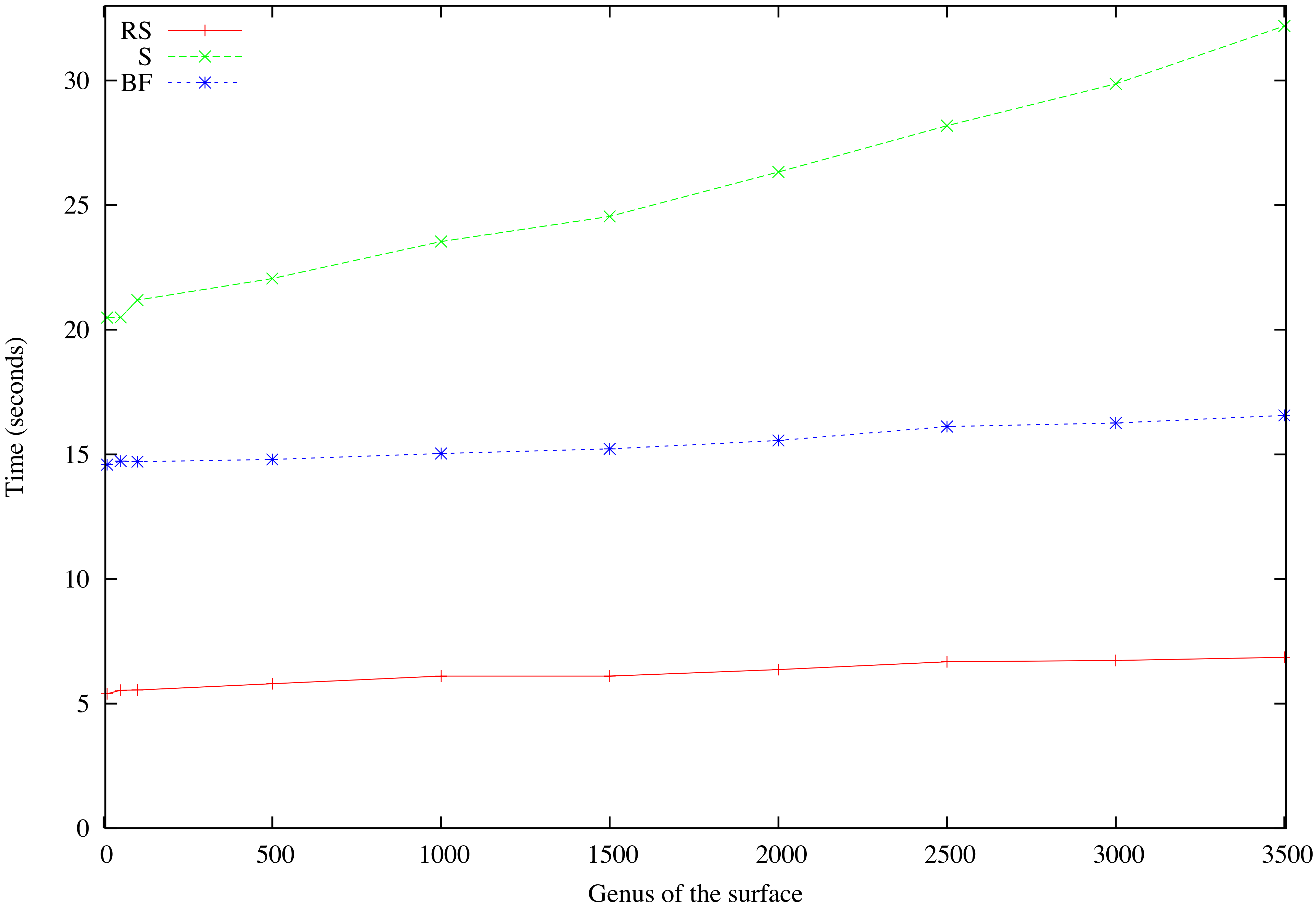
1e+06
1.5e+06
2e+06
2.5e+06
3e+06
3.5e+06
4e+06


RS
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Time (seconds)
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3000
3500
Genus of the surface

More Details

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Suneeta Ramaswami and Marcelo Siqueira

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CoRR, arXiv:1409.6015, 2014

Code:
`http://www.mat.ufrn.br/~mfsiqueira/Marcelo_Siqueiras_Web_Spot/Software.html`

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Questions?