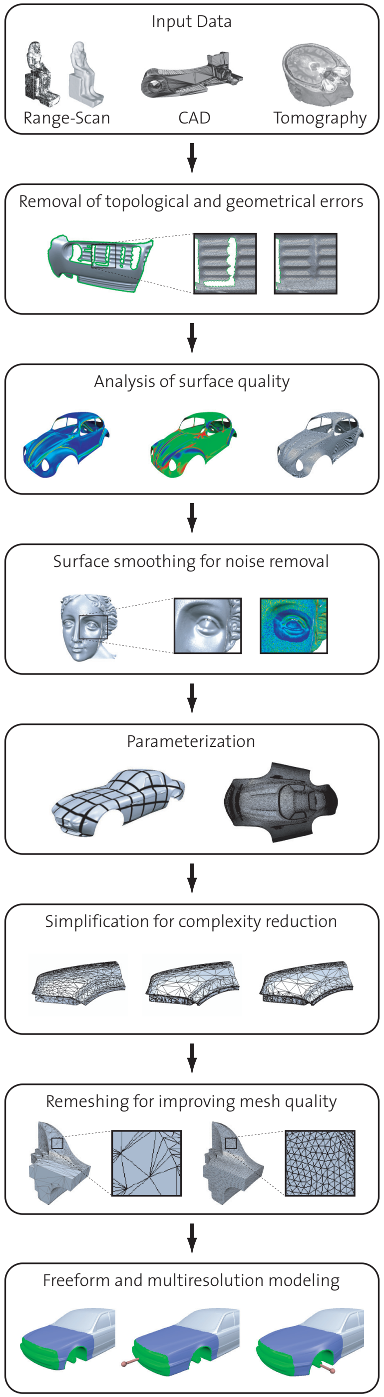

Surface Representations

Leif Kobbelt
RWTH Aachen University

Outline

- (mathematical) geometry representations
  - parametric vs. implicit
- approximation properties
- types of operations
  - distance queries
  - evaluation
  - modification / deformation
- data structures

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Mathematical Representations

- parametric
  - range of a function
  - surface patch

$$\mathbf{f} : R^2 \rightarrow R^3, \quad \mathcal{S}_\Omega = \mathbf{f}(\Omega)$$

- implicit
  - kernel of a function
  - level set

$$F : R^3 \rightarrow R, \quad \mathcal{S}_c = \{\mathbf{p} : F(\mathbf{p}) = c\}$$

2D-Example: Circle

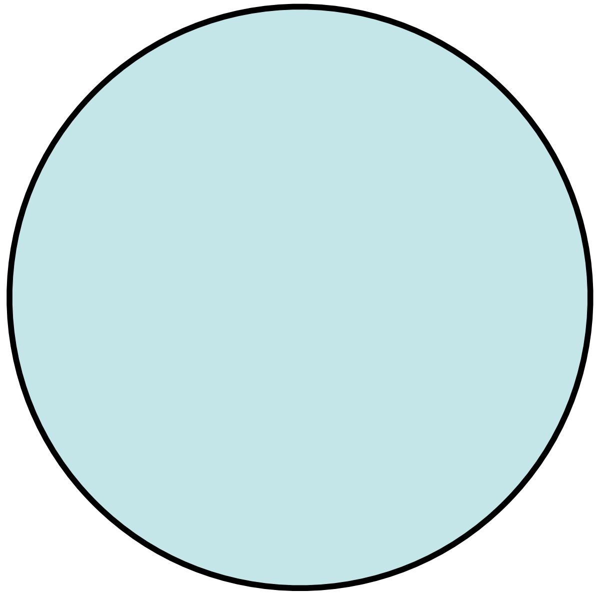
- parametric

$$\mathbf{f} : t \mapsto \begin{pmatrix} r\cos(t) \\ r\sin(t) \end{pmatrix}, \quad \mathcal{S} = \mathbf{f}([0, 2\pi])$$

- implicit

$$F(x, y) = x^2 + y^2 - r^2$$

$$\mathcal{S} = \{(x, y) : F(x, y) = 0\}$$



2D-Example: Island

- parametric

$$\mathbf{f} : t \mapsto \begin{pmatrix} ??? \\ ??? \end{pmatrix}, \qquad \mathcal{S} = \mathbf{f}([0, 2\pi])$$

- implicit

$$F(x, y) = \qquad ???$$

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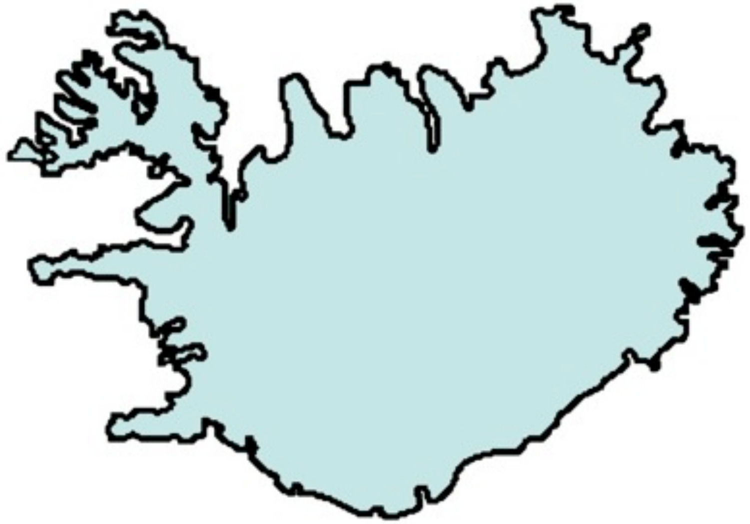
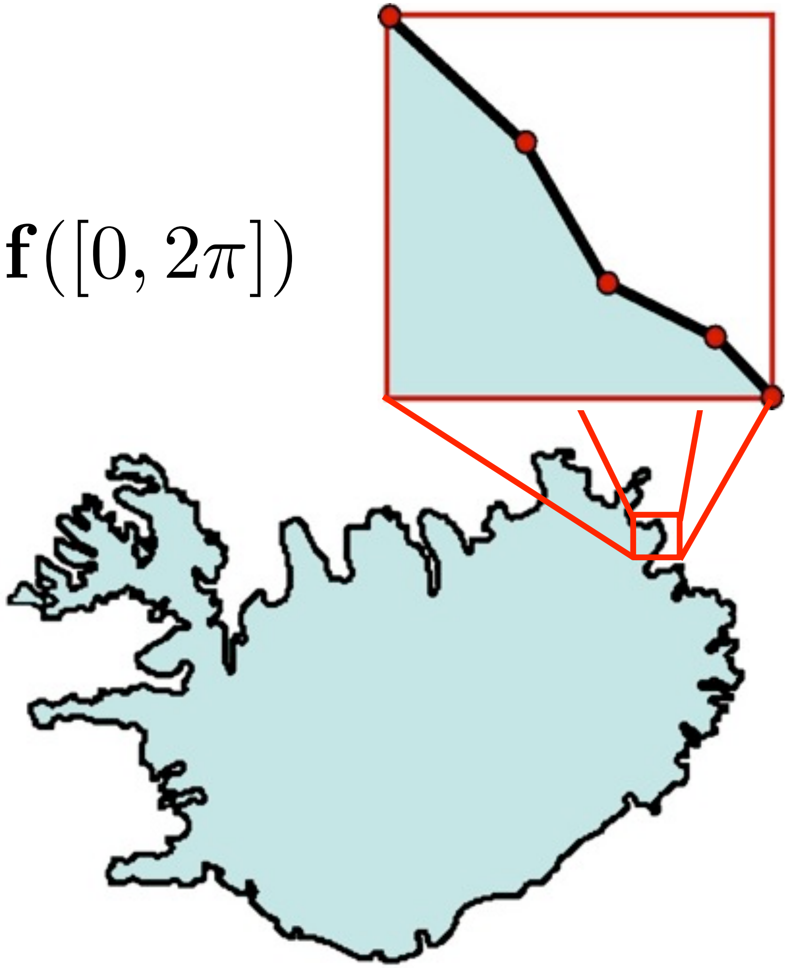


Approximation Quality

- ***piecewise*** parametric

$$\mathbf{f} : t \mapsto \begin{pmatrix} ??? \\ ??? \end{pmatrix}, \qquad \mathcal{S} = \mathbf{f}([0, 2\pi])$$



- piecewise implicit

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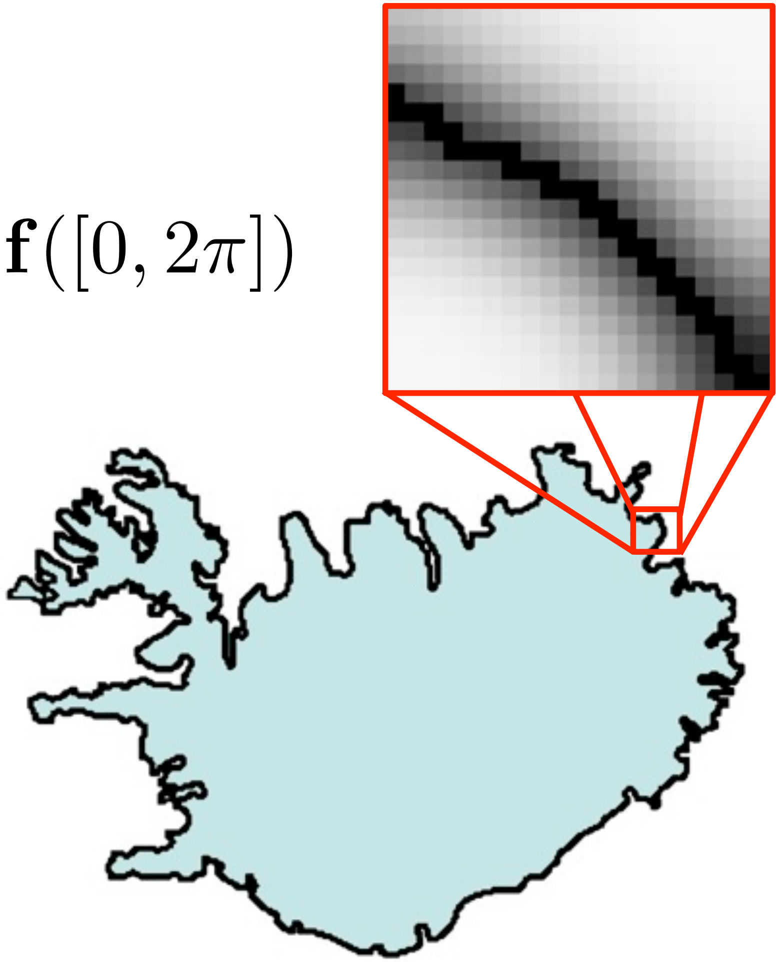

- ***piecewise*** implicit

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Requirements / Properties

- continuity
  - interpolation / approximation $\mathbf{f}(u_i, v_i) \approx \mathbf{p}_i$
- topological consistency
  - manifold-ness
- smoothness
  - C^0, C^1, C^2, ... C^k
- fairness
  - curvature distribution

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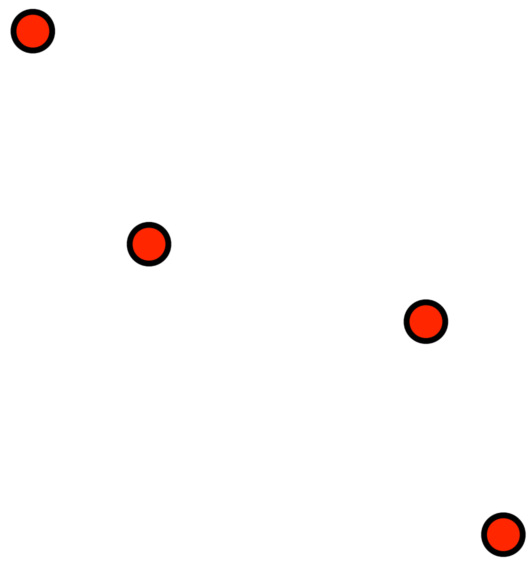

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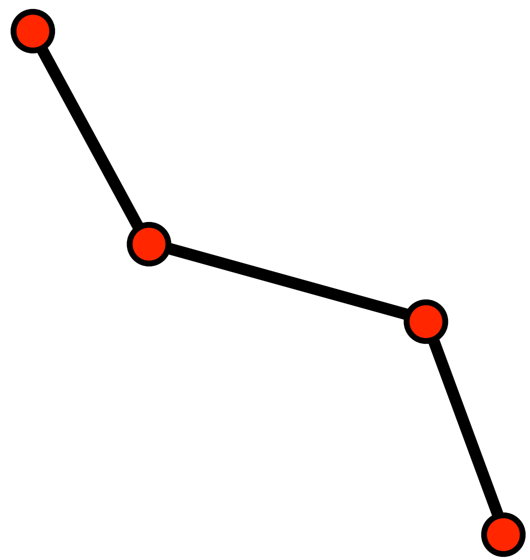

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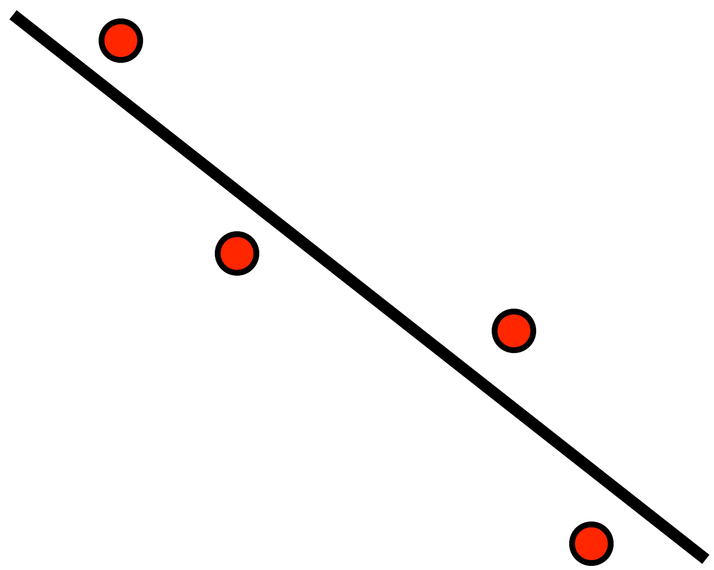

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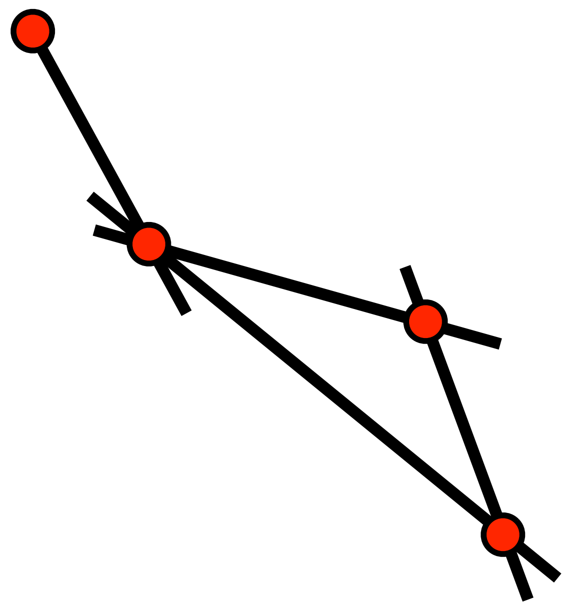
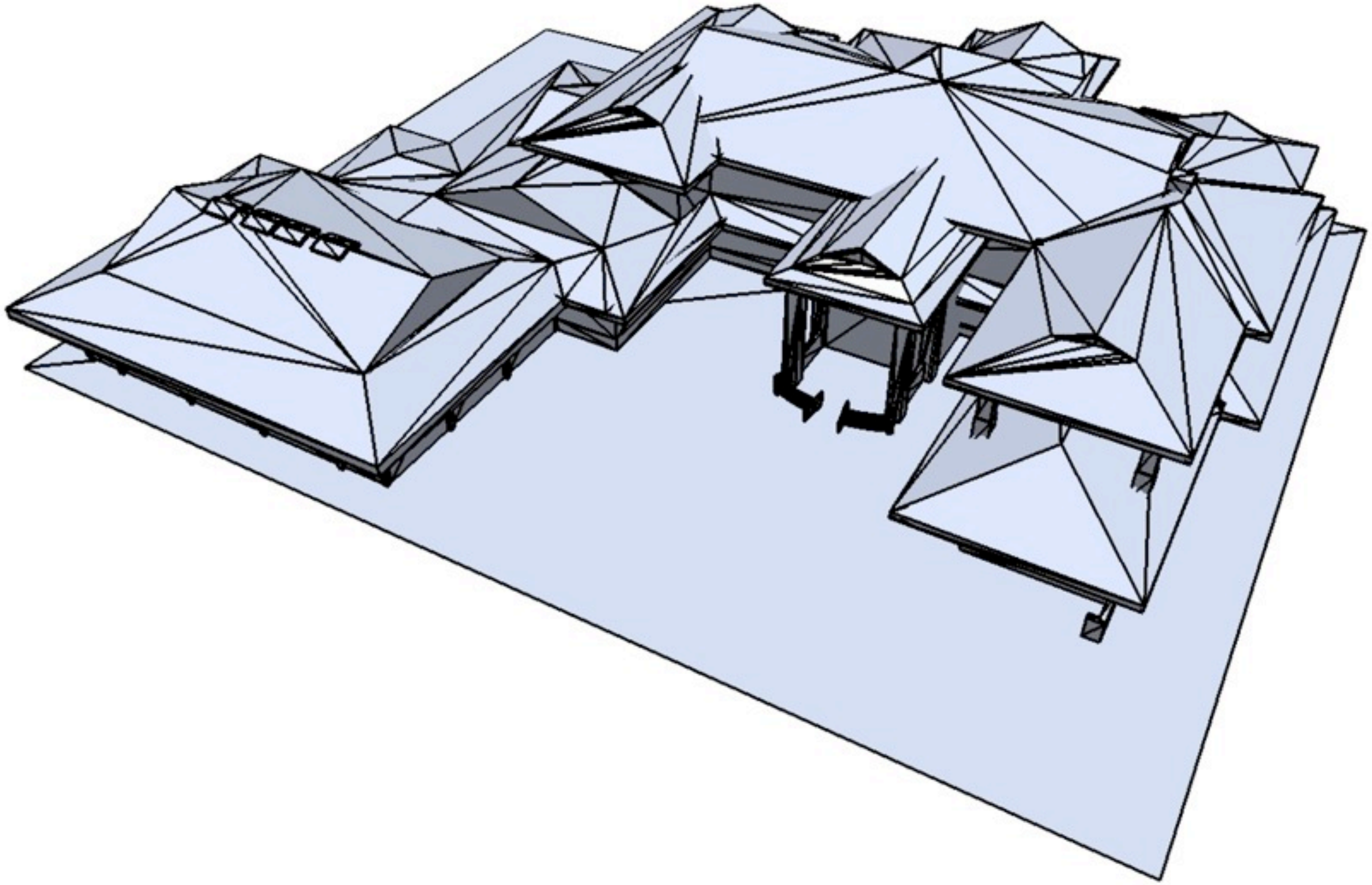


Topological Consistency



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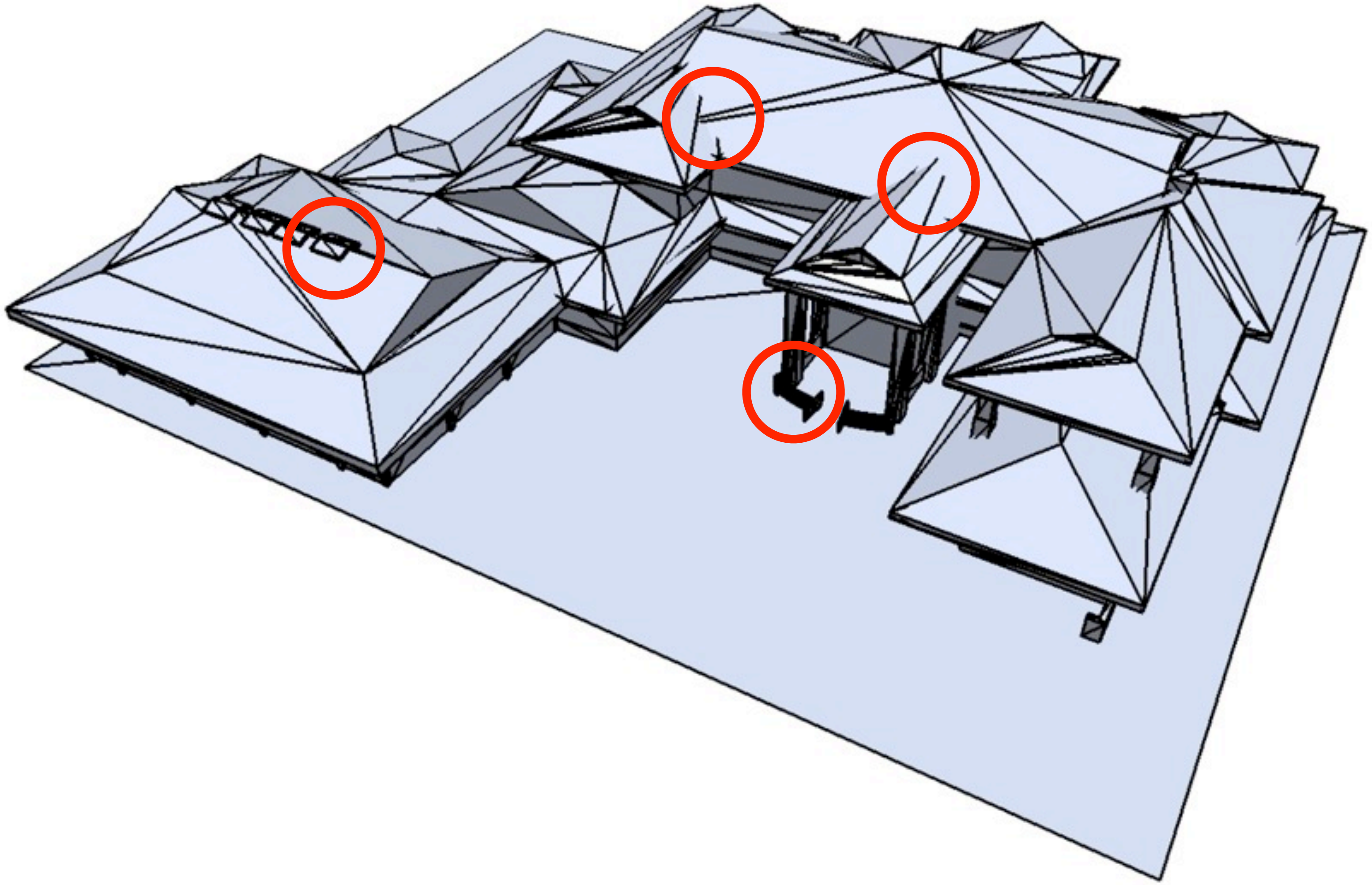

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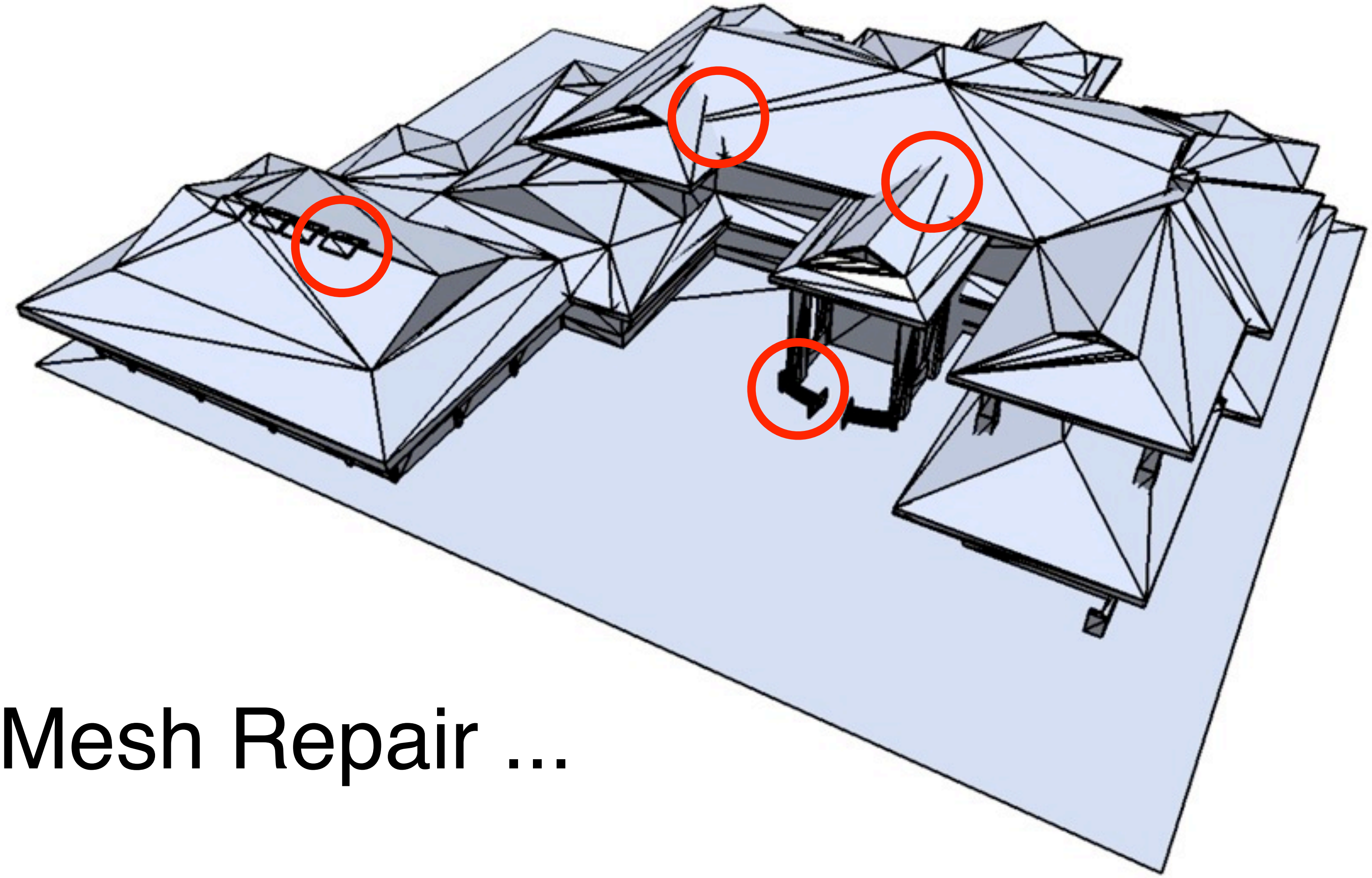

Mesh Repair ...

Closed 2-Manifolds

- parametric
  - disk-shaped neighborhoods
  - $\mathbf{f}(D_\varepsilon[u,v]) = D_\delta[\mathbf{f}(u,v)]$ + injectivity
- implicit
  - surface of a “physical” solid
  - $F(x,y,z) = c, \quad \|\nabla F(x,y,z)\| \neq 0$

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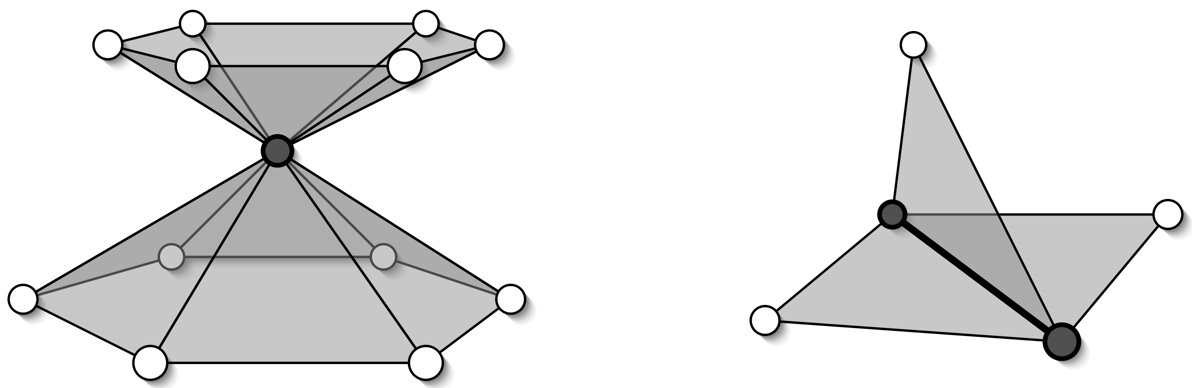

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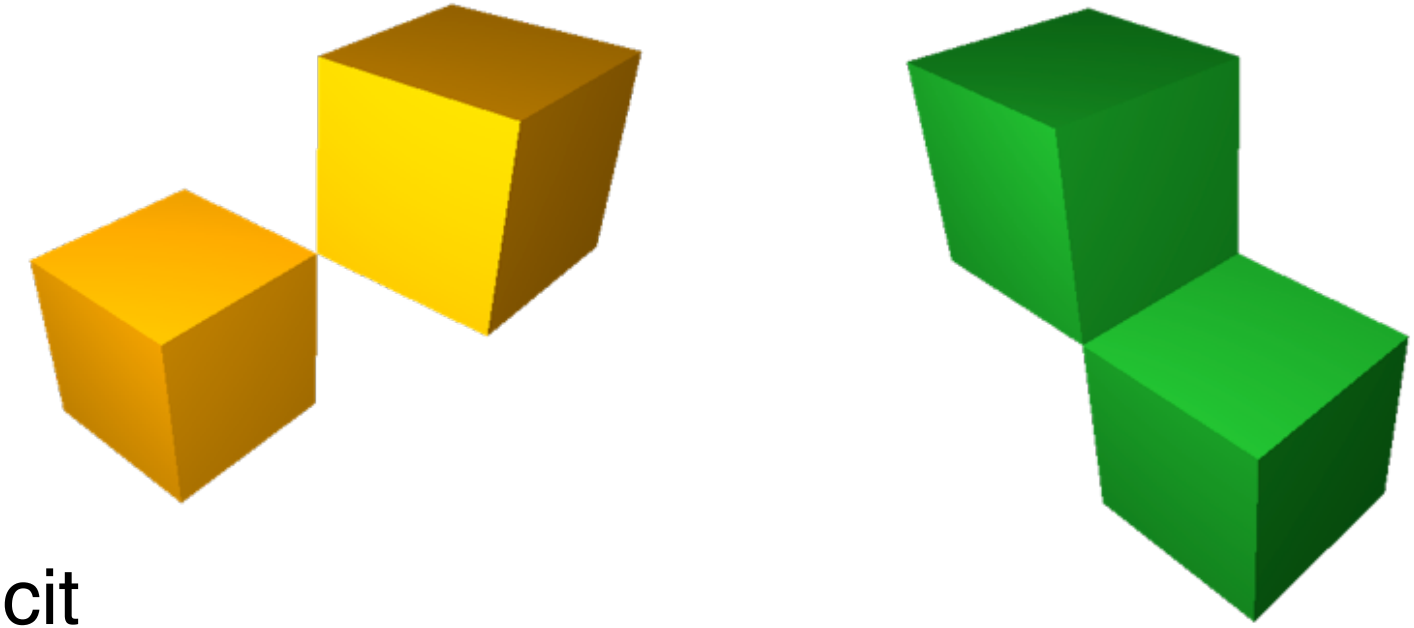

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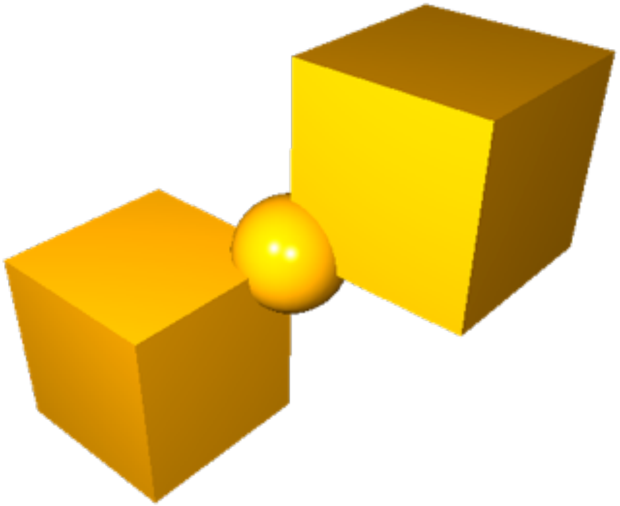

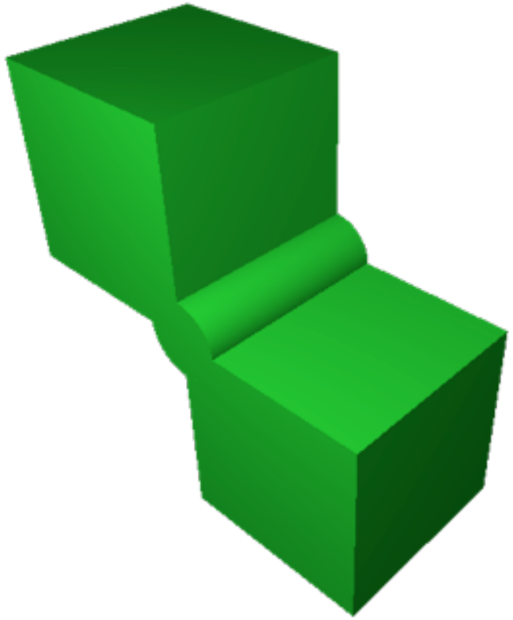

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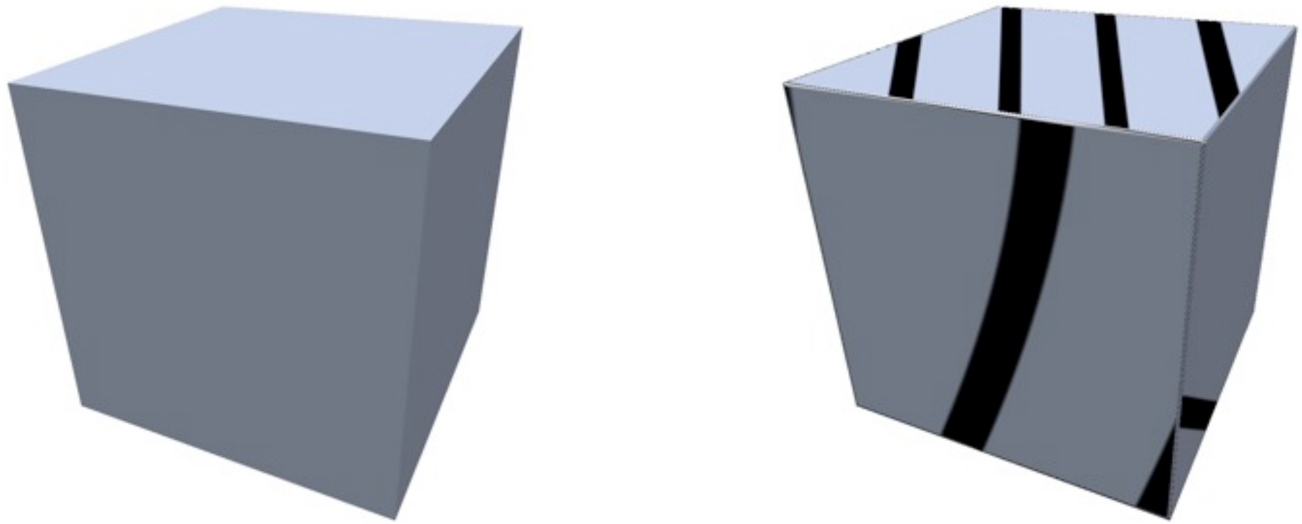
Smoothness

- position continuity : C^0
- tangent continuity : C^1
- curvature continuity : C^2



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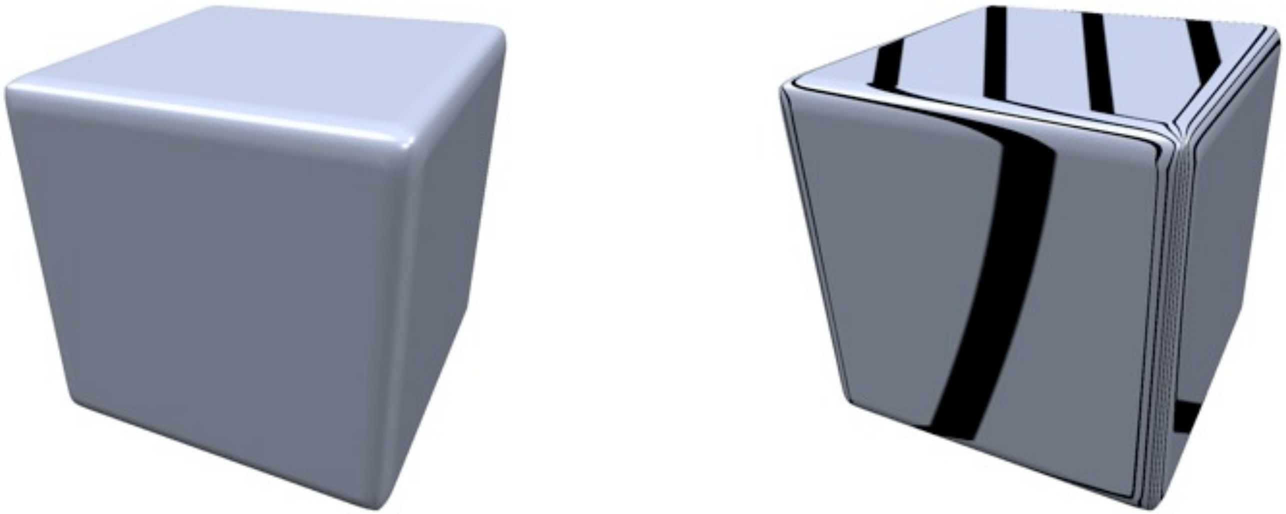

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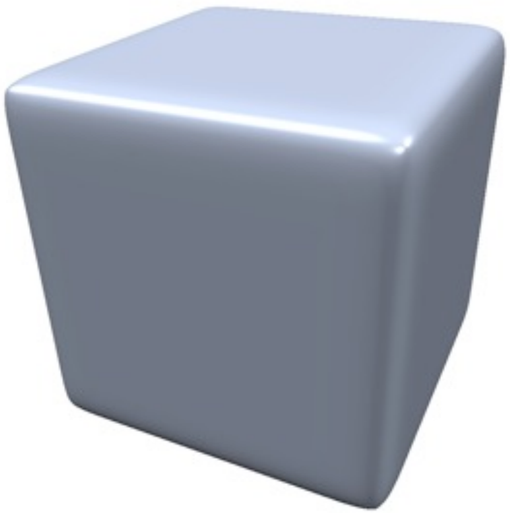
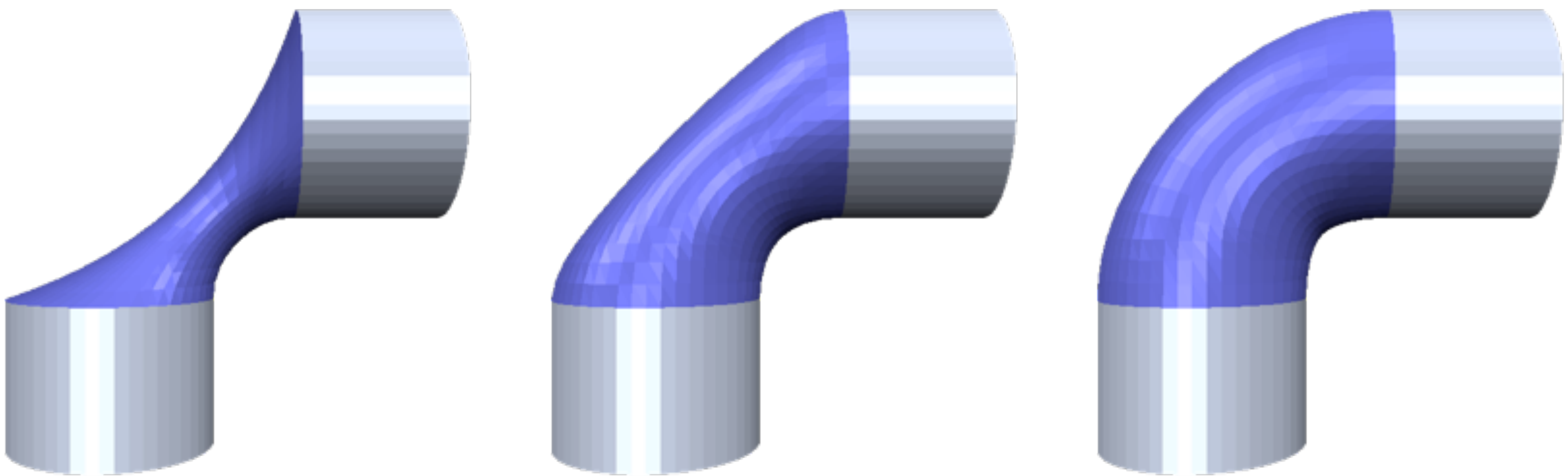


Fairness

- minimum surface area
- minimum curvature
- minimum curvature variation



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- (mathematical) geometry representations
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Polynomials

- computable functions

$$\mathbf{p}(t) \;=\; \sum_{i=0}^{p} \mathbf{c}_i \, t^i \;=\; \sum_{i=0}^{p} \mathbf{c}'_i \, \Phi_i(t)$$

- Taylor expansion

$$\mathbf{f}(h) \;=\; \sum_{i=0}^{p} \frac{1}{i!} \mathbf{f}^{(i)}(0) \, h^i \;+\; O(h^{p+1})$$

- interpolation error (mean value theorem)

$$\mathbf{p}(t_i) = \mathbf{f}(t_i), \quad t_i \in [0, h]$$

$$\|\mathbf{f}(t) - \mathbf{p}(t)\| \;=\; \frac{1}{(p+1)!} \mathbf{f}^{(p+1)}(t^*) \prod_{i=0}^{p} (t - t_i) \;=\; O(h^{(p+1)})$$

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Implicit Polynomials

- interpolation error of the function values

$$\|F(x,y,z) - P(x,y,z)\| \;=\; O(h^{(p+1)})$$

- approximation error of the contour

$$\triangle\mathbf{p} = \lambda\,\nabla F(\mathbf{p}) \qquad \frac{F(\mathbf{p}+\triangle\mathbf{p}) - F(\mathbf{p})}{\|\triangle\mathbf{p}\|} \;\approx\; \|\nabla F(\mathbf{p})\|$$

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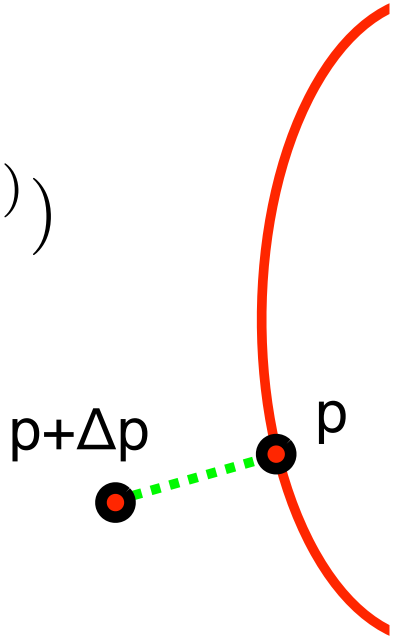

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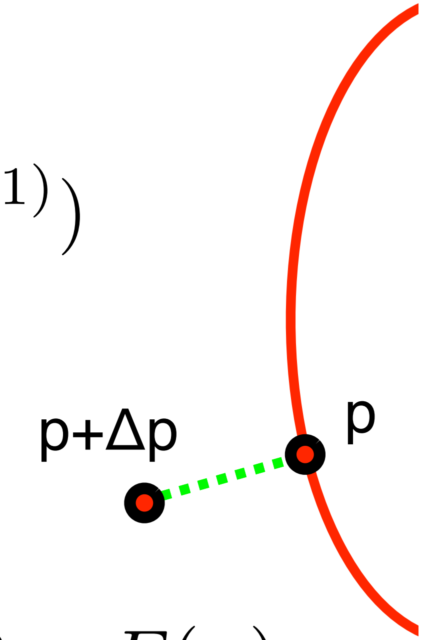

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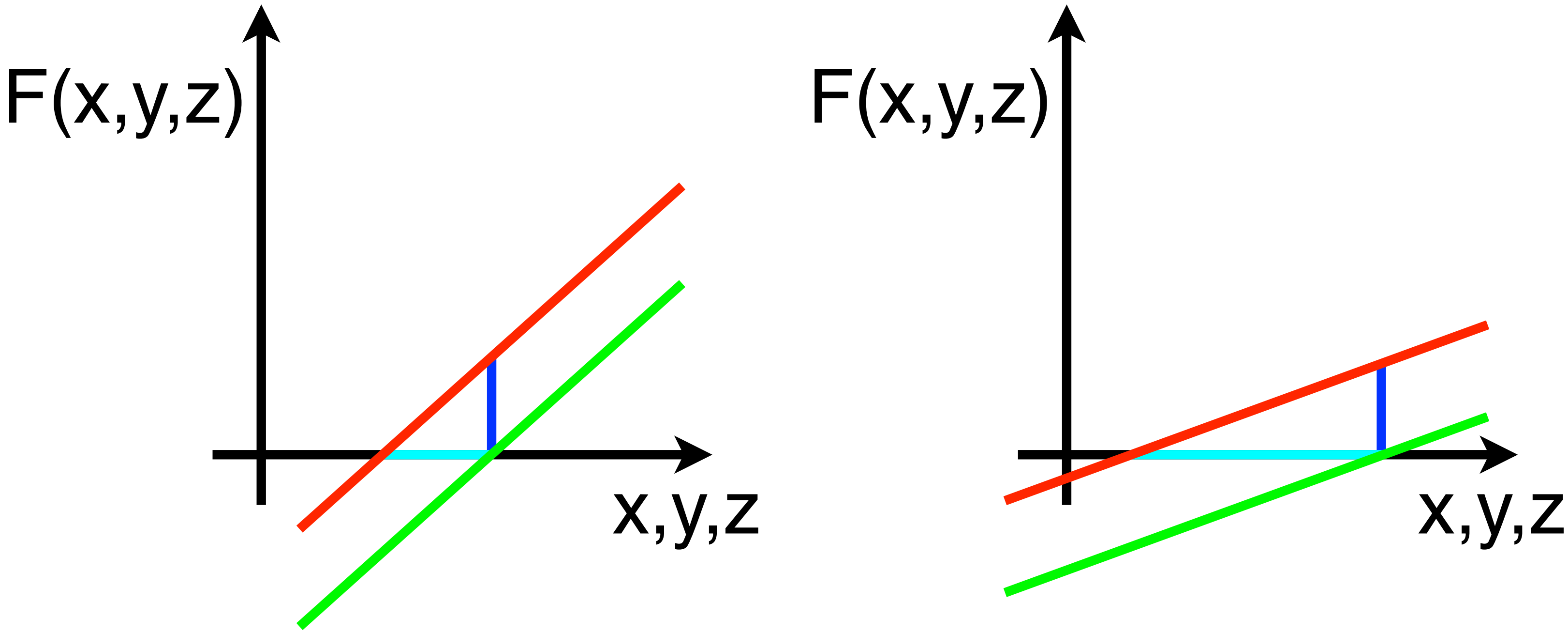
(gradient bounded from below)

Implicit Polynomials



large
gradient

small
gradient

Polynomial Approximation

- approximation error is **O(h^{p+1})**
- improve approximation quality by
  - increasing **p** ... higher order polynomials
  - decreasing **h** ... smaller / more segments
- issues
  - smoothness of the target data ($\max_t f^{(p+1)}(t)$)
  - handling higher order patches (e.g. boundary conditions)

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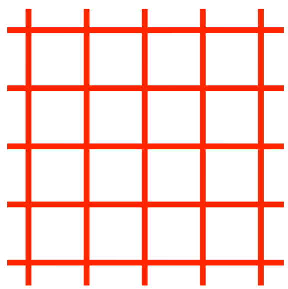
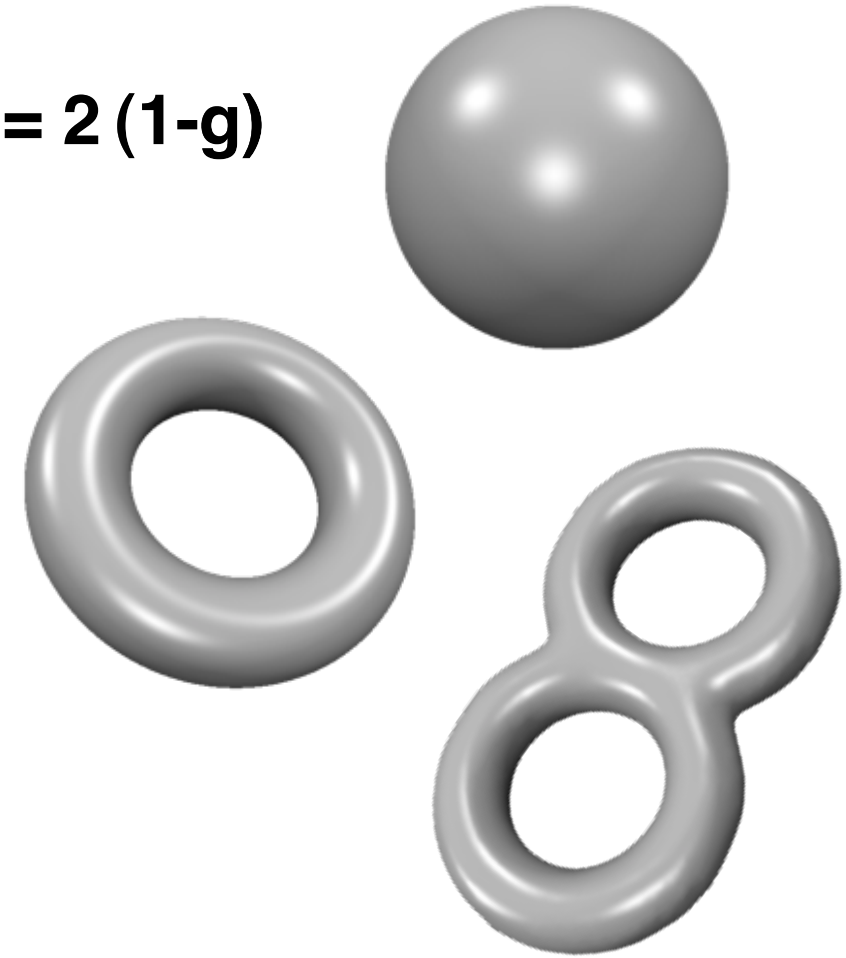
MOTD: p = 1

Piecewise Definition

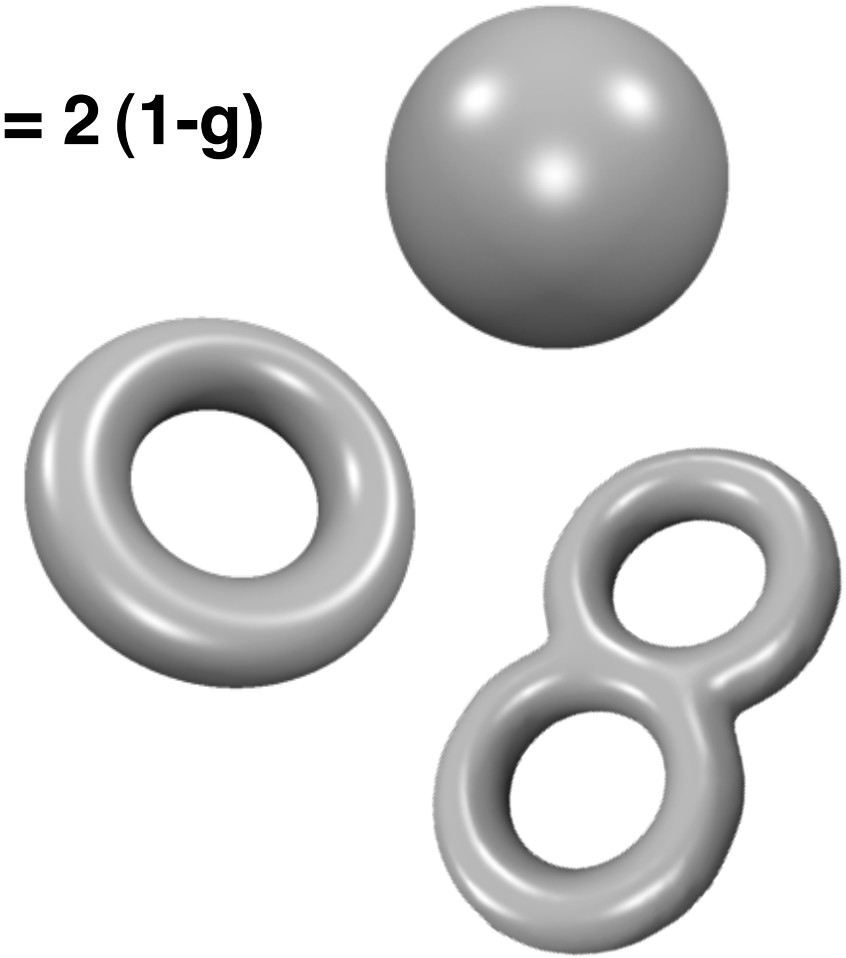
- parametric
  - Euler formula: $\mathbf{V - E + F = 2\,(1\text{-}g)}$
  - regular **quad** meshes
    - $F \approx V$
    - $E \approx 2V$
    - average valence = 4
  - quasi-regular
  - semi-regular

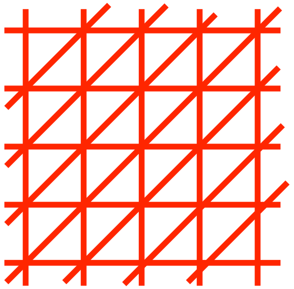




Piecewise Definition

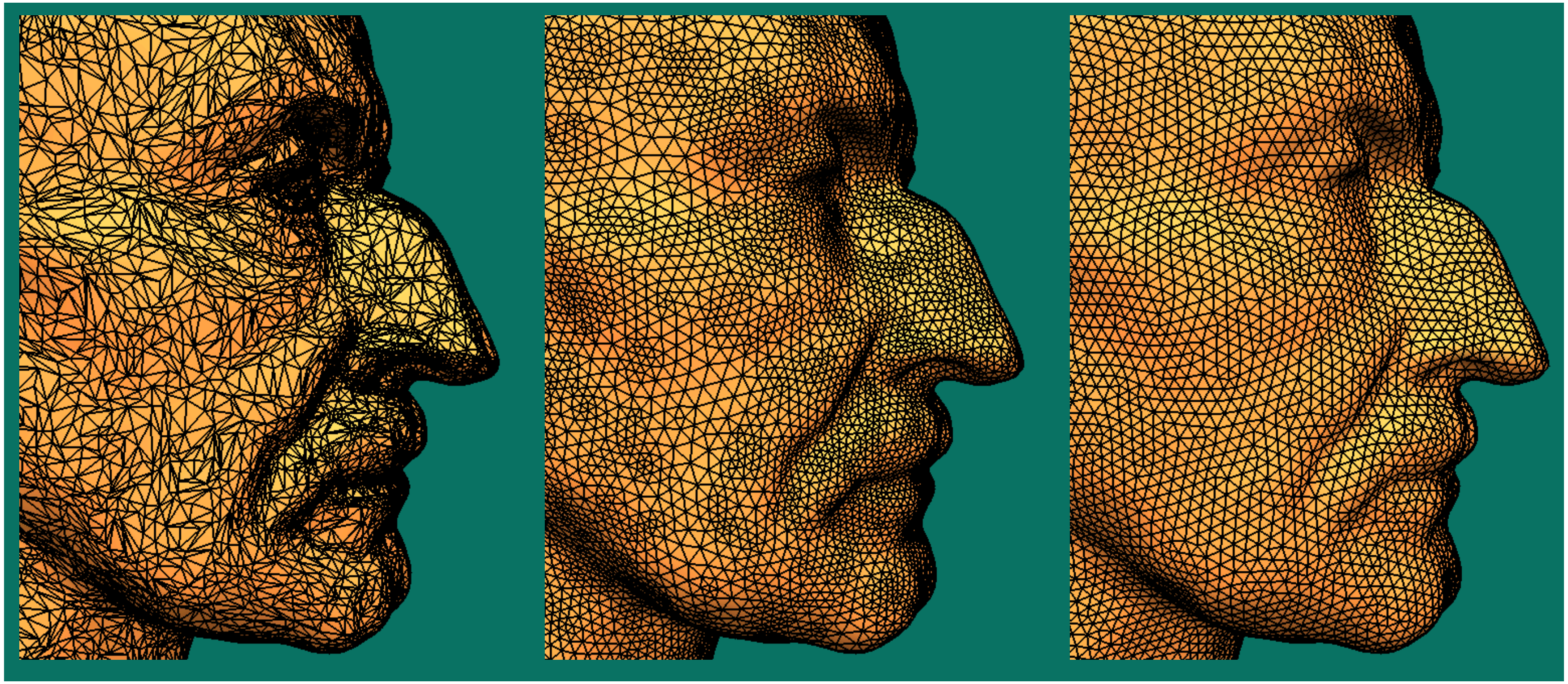
- parametric
  - Euler formula: **V - E + F = 2 (1-g)**
  - regular **triangle** meshes
    - $F \approx 2V$
    - $E \approx 3V$
    - average valence = 6
  - quasi-regular
  - semi-regular





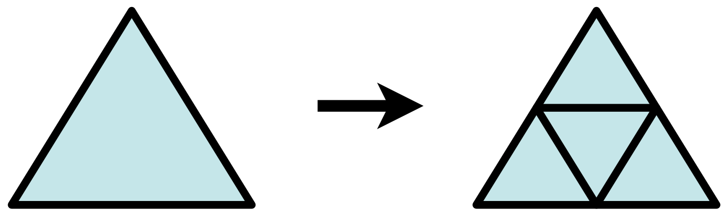
Piecewise Definition

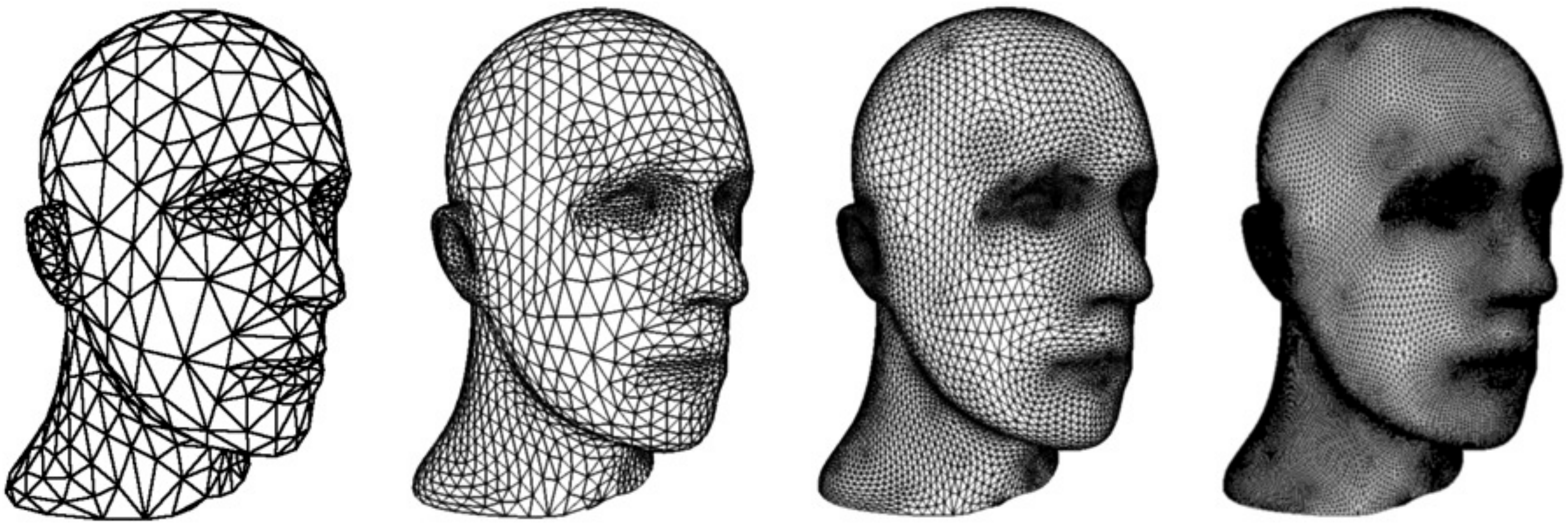
- quasi regular (→ remeshing, Pierre)



Piecewise Definition

- semi regular





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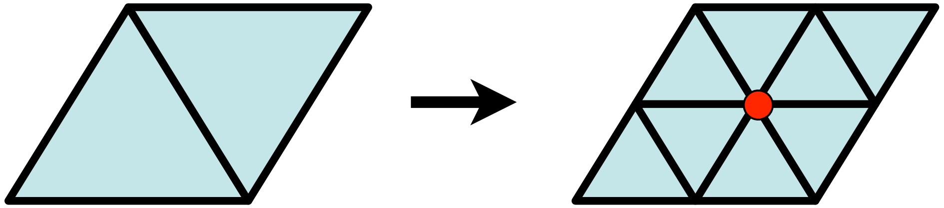
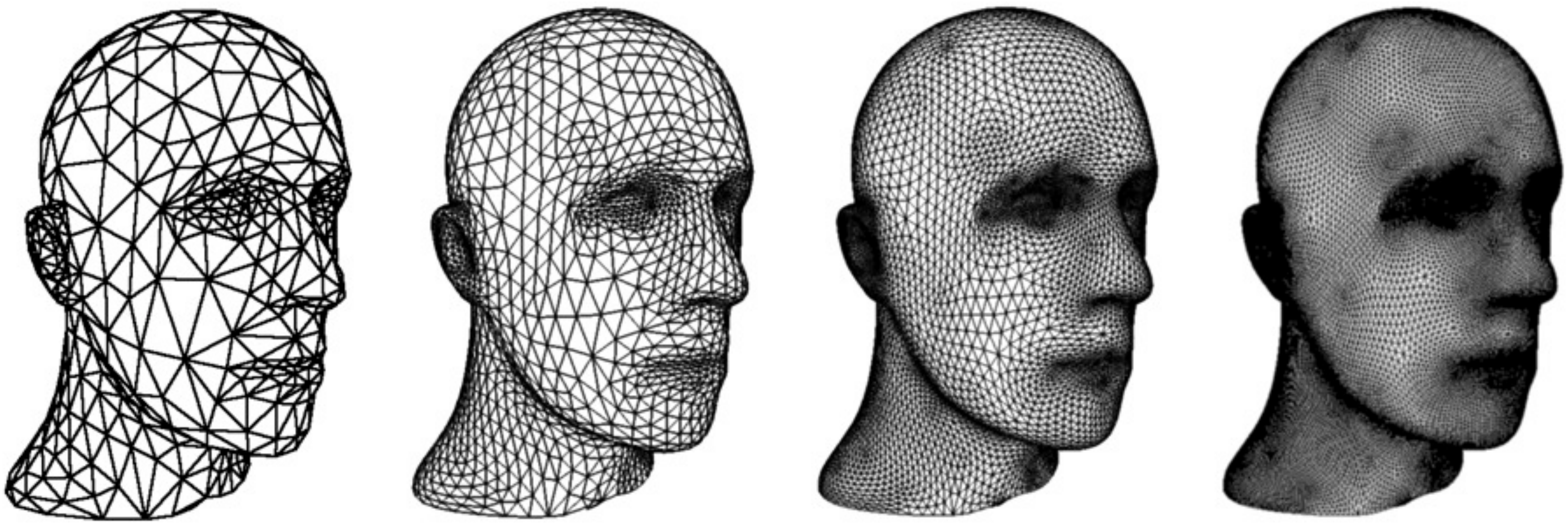
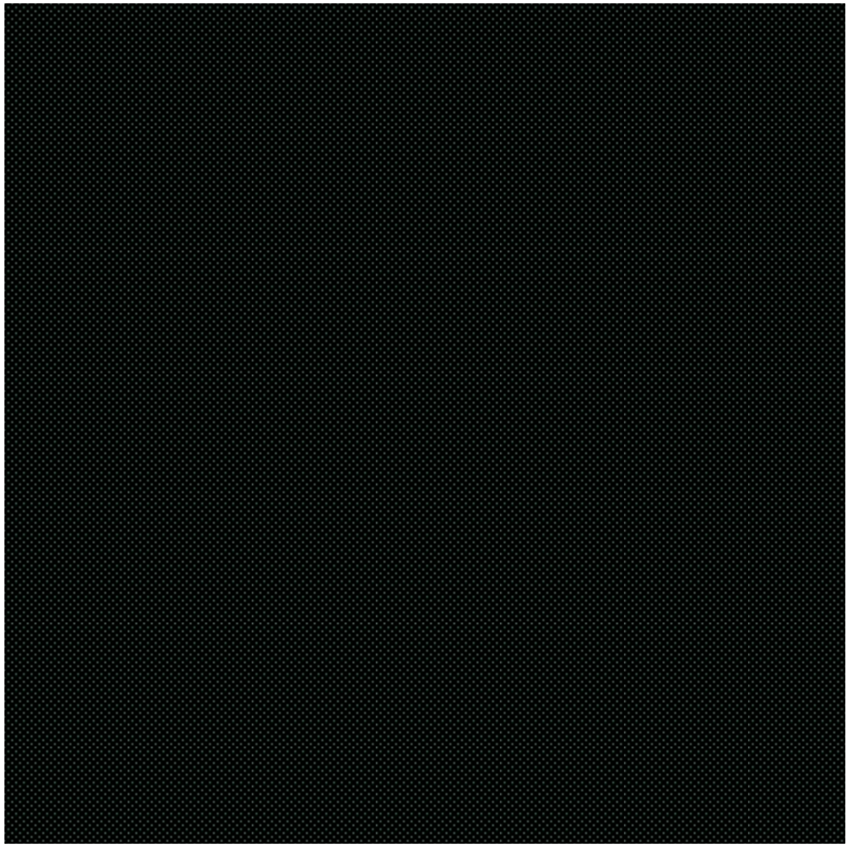




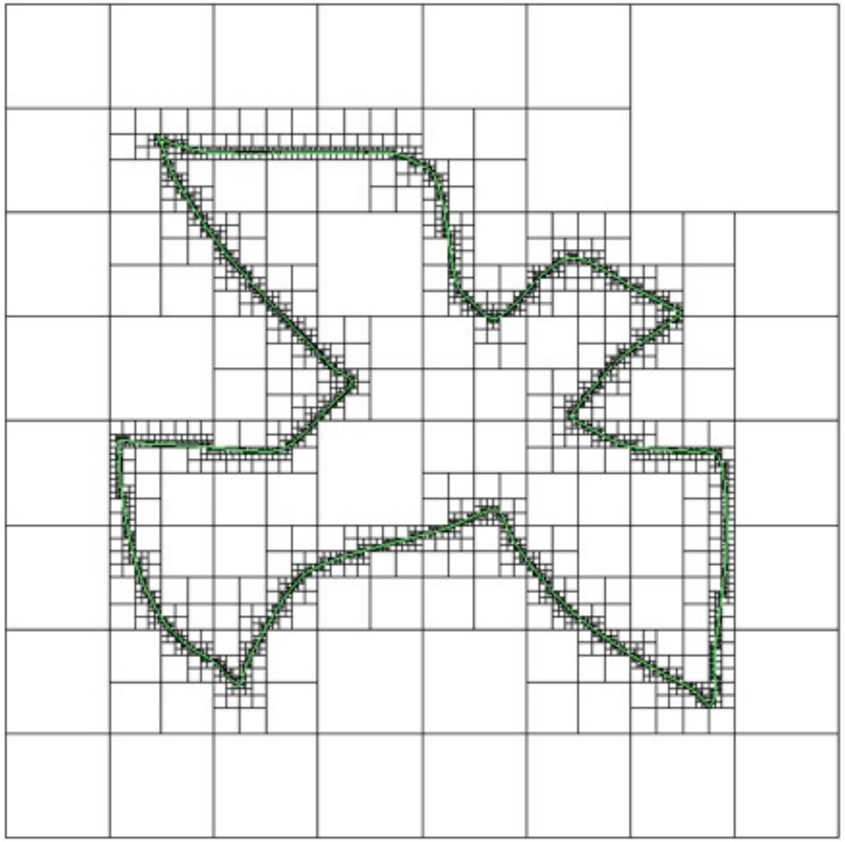
Piecewise Definition

- implicit
  - regular voxel grids $O(h^{-3})$
  - three color octrees
    - surface-adaptive refinement $O(h^{-2})$
    - feature-adaptive refinement $O(h^{-1})$
  - irregular hierarchies
    - binary space partition $O(h^{-1})$
      (BSP)

3-Color Octree

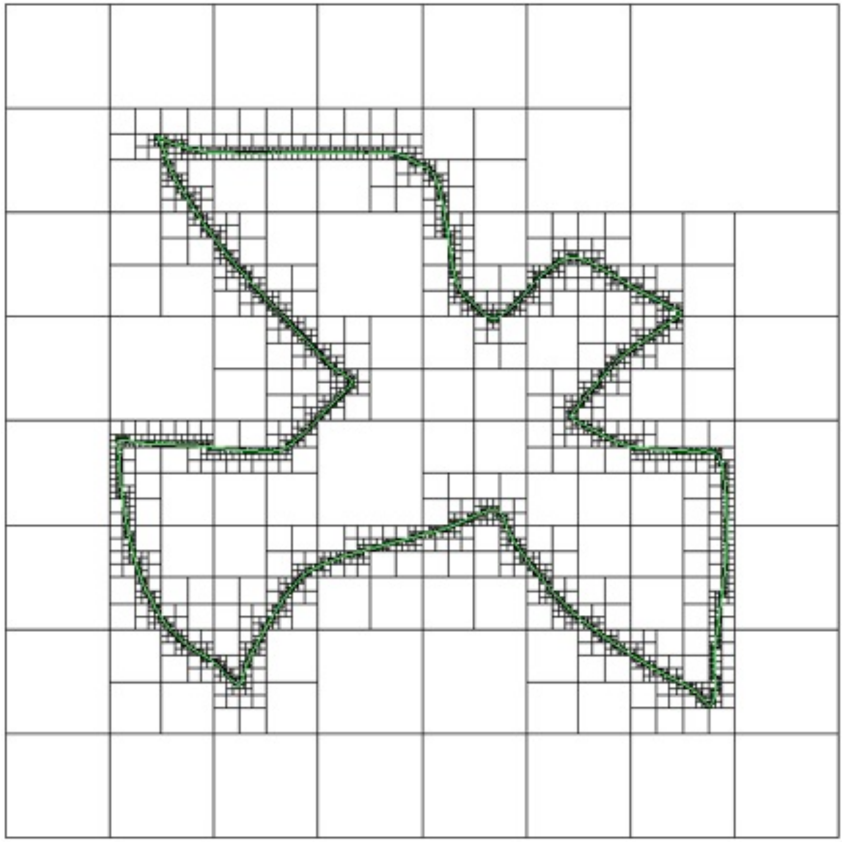


1048576 cells



12040 cells

Adaptively Sampled Distance Fields



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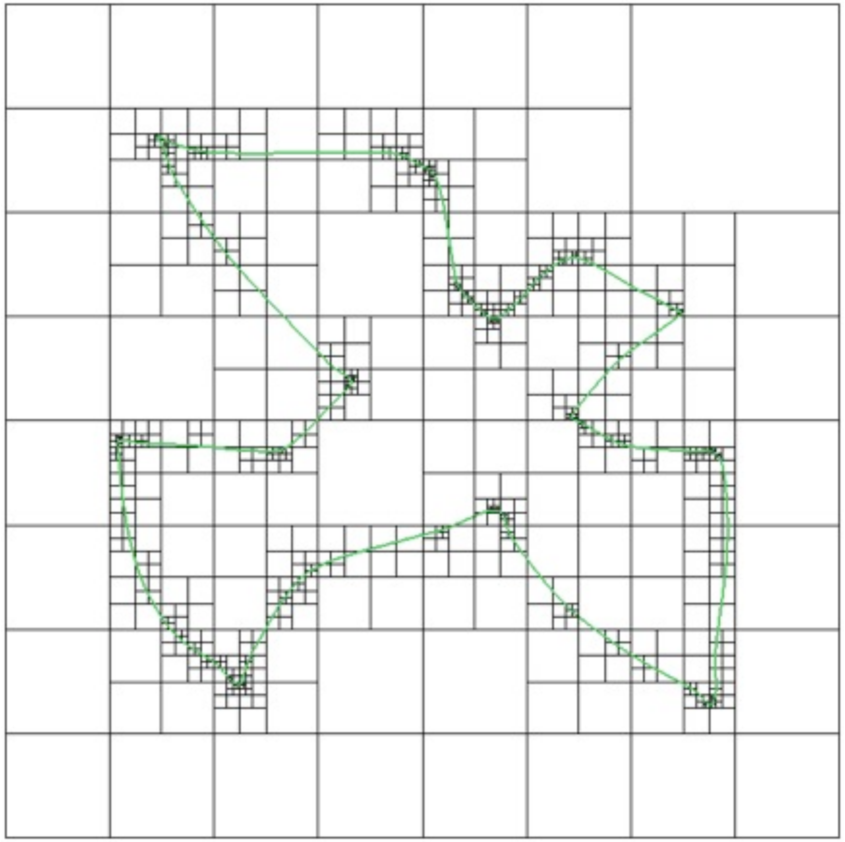
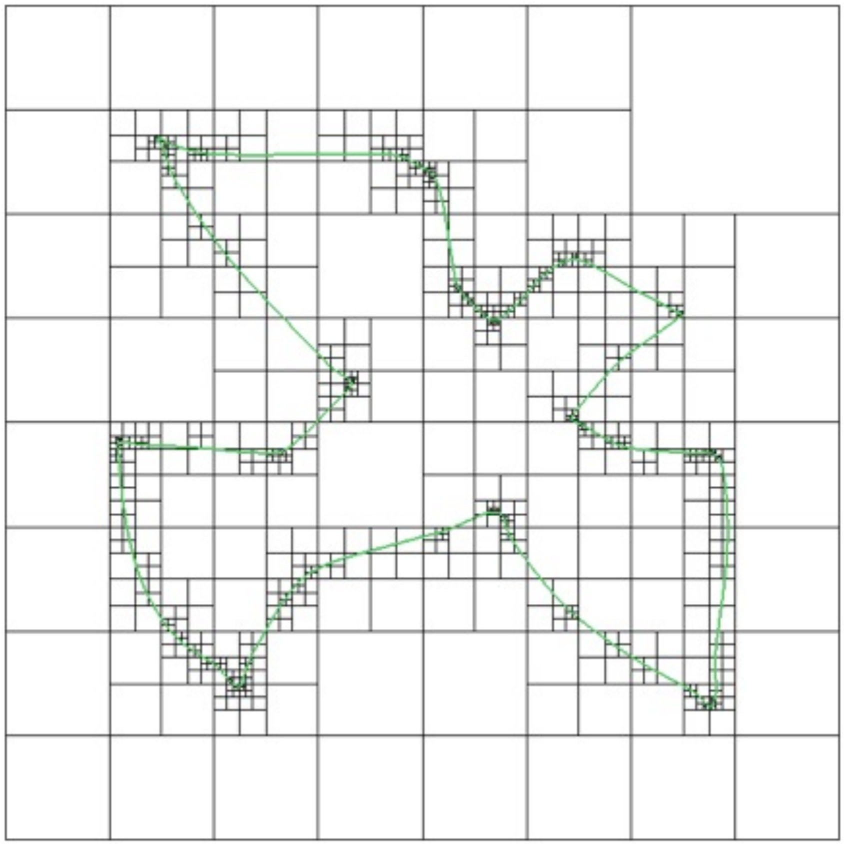


895 cells

Binary Space Partitions



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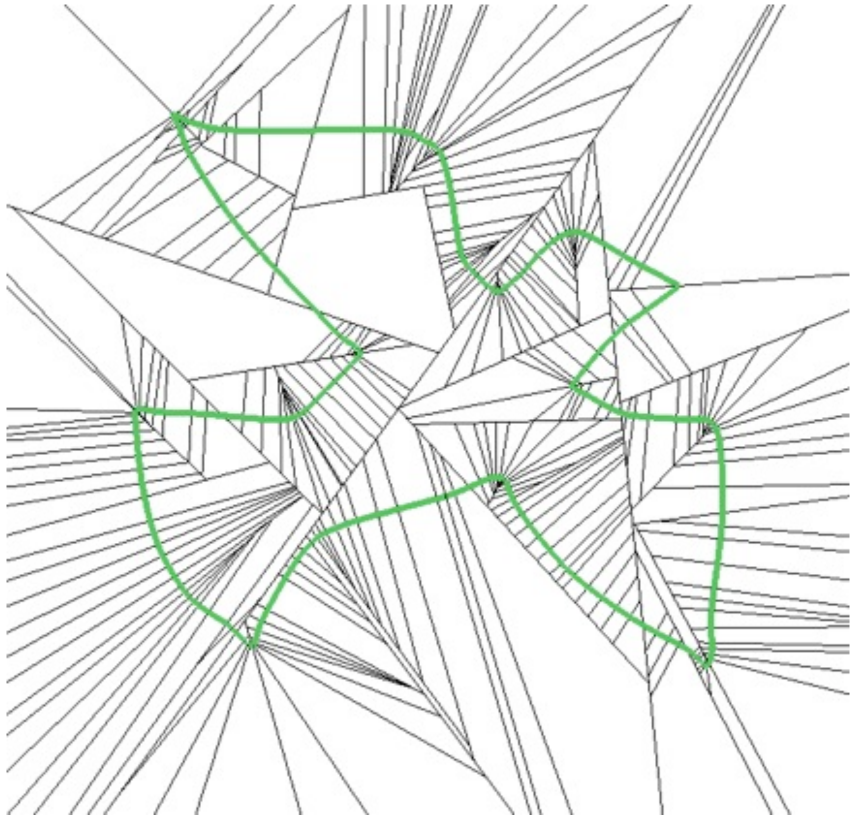
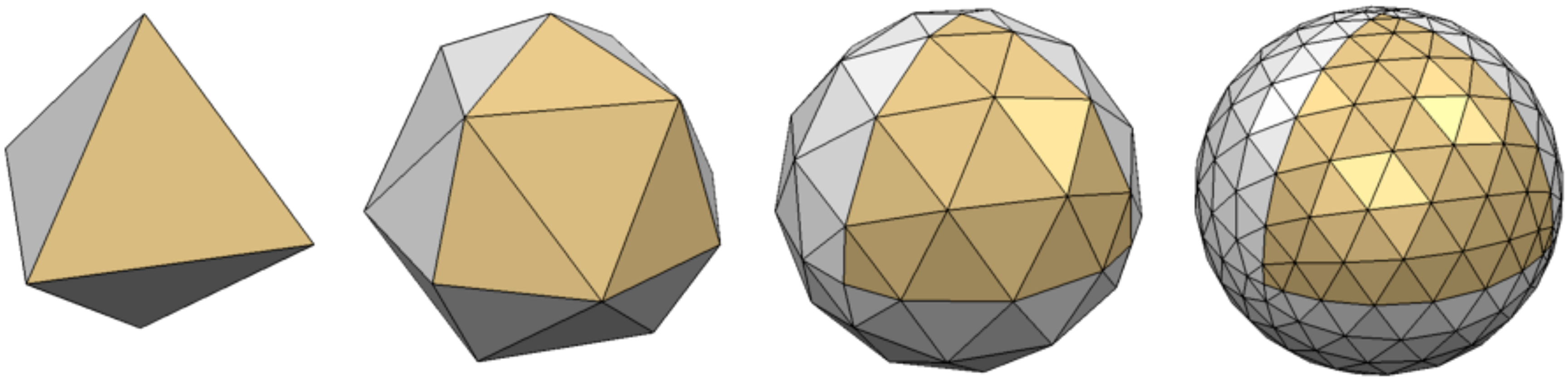


254 cells

Message of the Day ...

- polygonal meshes are a good compromise
  - approximation $O(h^2)$ ... error * #faces = const.
  - arbitrary topology
  - flexibility for piecewise smooth surfaces
  - flexibility for adaptive refinement



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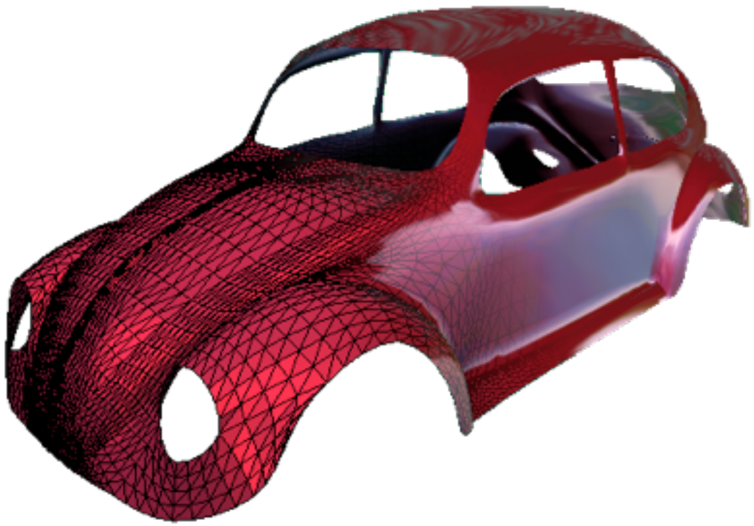

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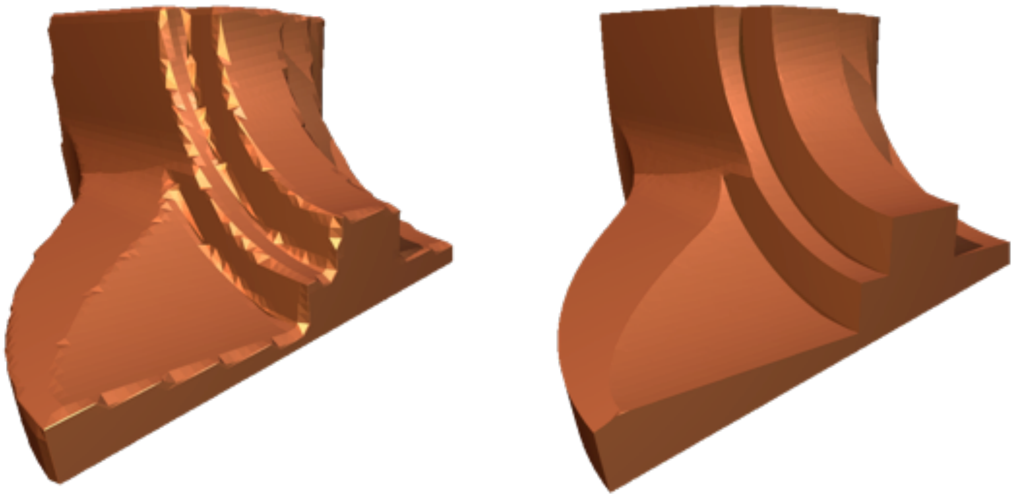

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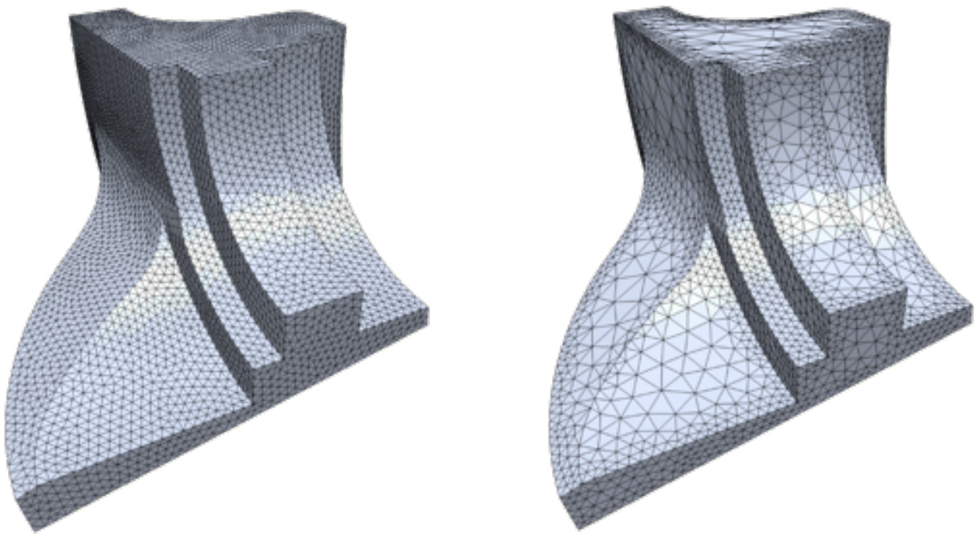

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  - approximation $O(h^2)$ ... error * #faces = const.
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  - flexibility for piecewise smooth surfaces
  - flexibility for adaptive refinement
  - efficient rendering

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- polygonal meshes are a good compromise
  – approximation $O(h^2)$ ... error * #faces = const.
  – arbitrary topology
  – flexibility for piecewise smooth surfaces
  – flexibility for adaptive refinement
  – efficient rendering

- implicit representation can support efficient access to vertices, faces, ...

Outline

- (mathematical) geometry representations
  - parametric vs. implicit
- approximation properties
- types of operations
  - evaluation
  - distance queries
  - modification / deformation
- data structures

Evaluation

- smooth parametric surfaces
  - positions $\mathbf{f}(u,v)$
  - normals $\mathbf{n}(u,v) = \mathbf{f}_u(u,v) \times \mathbf{f}_v(u,v)$
  - curvatures $\mathbf{c}(u,v) = C\Big(\mathbf{f}_{uu}(u,v), \mathbf{f}_{uv}(u,v), \mathbf{f}_{vv}(u,v)\Big)$

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- generalization to triangle meshes
  - positions (barycentric coordinates)

$$(\alpha,\beta) \mapsto \alpha\,\mathbf{P}_1 + \beta\,\mathbf{P}_2 + (1-\alpha-\beta)\,\mathbf{P}_3$$

$$0 \le \alpha, \quad 0 \le \beta, \quad \alpha + \beta \le 1$$

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$$(\alpha, \beta, \gamma) \;\mapsto\; \alpha\,\mathbf{P}_1 + \beta\,\mathbf{P}_2 + \gamma\,\mathbf{P}_3$$

$$\alpha + \beta + \gamma \;=\; 1$$

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- generalization to triangle meshes
  - positions (barycentric coordinates)

$$\alpha\,\mathbf{u} + \beta\,\mathbf{v} + \gamma\,\mathbf{w} \;\mapsto\; \alpha\,\mathbf{P}_1 + \beta\,\mathbf{P}_2 + \gamma\,\mathbf{P}_3$$

$$\alpha + \beta + \gamma \;=\; 0$$

Evaluation

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→ Parametrization
Bruno

$$\alpha\,\mathbf{u} + \beta\,\mathbf{v} + \gamma\,\mathbf{w} \mapsto \alpha\,\mathbf{P}_1 + \beta\,\mathbf{P}_2 + \gamma\,\mathbf{P}_3$$

$$\alpha + \beta + \gamma = 0$$

Evaluation

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  - positions $\mathbf{f}(u,v)$
  - normals $\mathbf{n}(u,v) = \mathbf{f}_u(u,v) \times \mathbf{f}_v(u,v)$
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- generalization to triangle meshes
  - positions (barycentric coordinates)
  - normals (per face, Phong)

$$\mathbf{N} = (\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_1)$$

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$$\alpha\,\mathbf{u} + \beta\,\mathbf{v} + \gamma\,\mathbf{w} \;\mapsto\; \alpha\,\mathbf{N}_1 + \beta\,\mathbf{N}_2 + \gamma\,\mathbf{N}_3$$

Evaluation

- smooth parametric surfaces
  - positions $\mathbf{f}(u,v)$
  - normals $\mathbf{n}(u,v) = \mathbf{f}_u(u,v) \times \mathbf{f}_v(u,v)$
  - curvatures $\mathbf{c}(u,v) = C\Big(\mathbf{f}_{uu}(u,v), \mathbf{f}_{uv}(u,v), \mathbf{f}_{vv}(u,v)\Big)$
- generalization to triangle meshes
  - positions (barycentric coordinates)
  - normals (per face, Phong)
  - curvatures ... (→ smoothing, Mark)

Distance Queries

- parametric
  - for smooth surfaces: find orthogonal base point

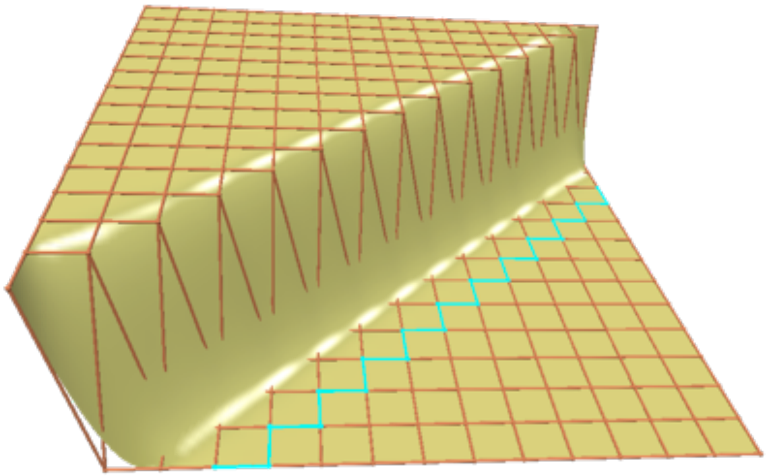
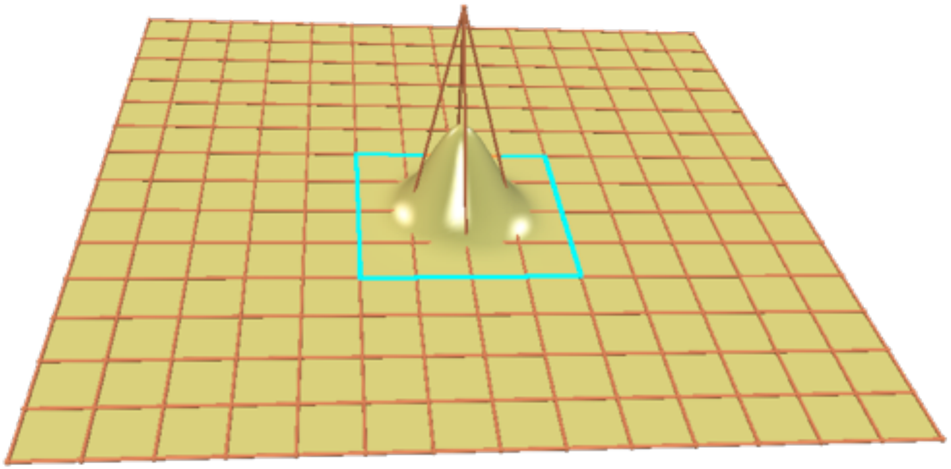
$$[\mathbf{p} - \mathbf{f}(u,v)] \times \mathbf{n}(u,v) \ = \ \mathbf{0}$$

  - for triangle meshes
    - use kd-tree or BSP to find closest triangle
    - find base point by Newton iteration (use Phong normal field)

Modifications

- parameteric
  - control vertices
  - free-form deformation
  - boundary constraint modeling

$$\mathbf{f}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{c}_{ij} N_i^n(u) N_j^m(v)$$

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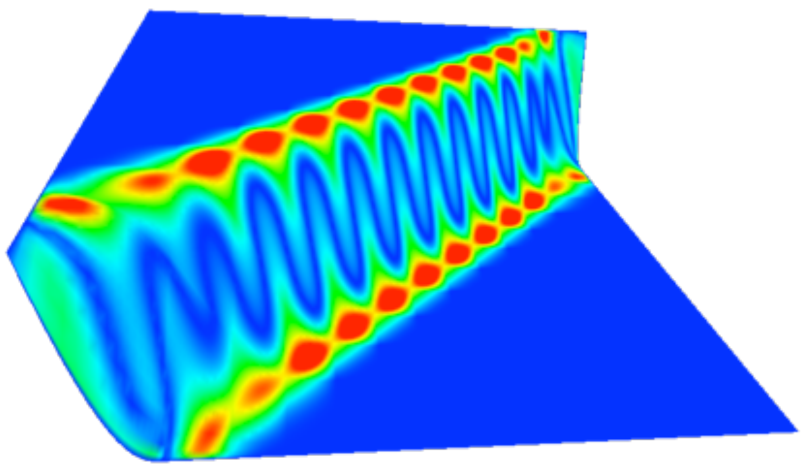

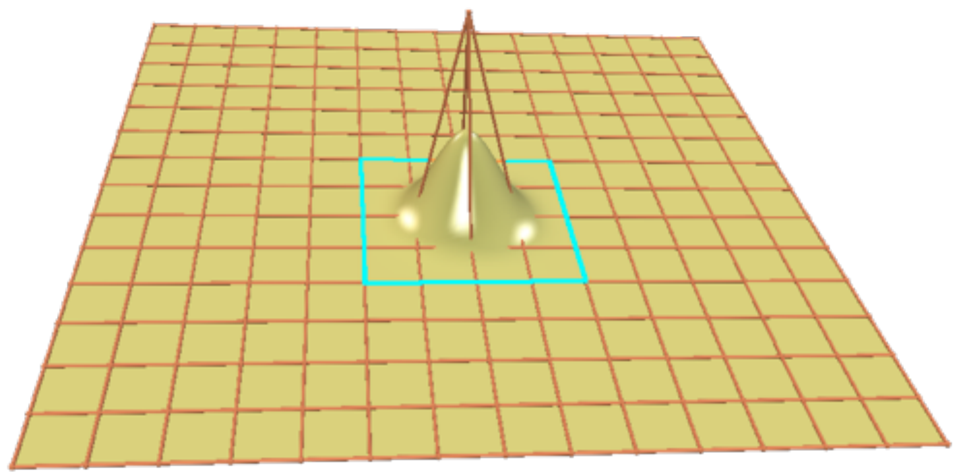

Modifications

- parameteric
  - control vertices
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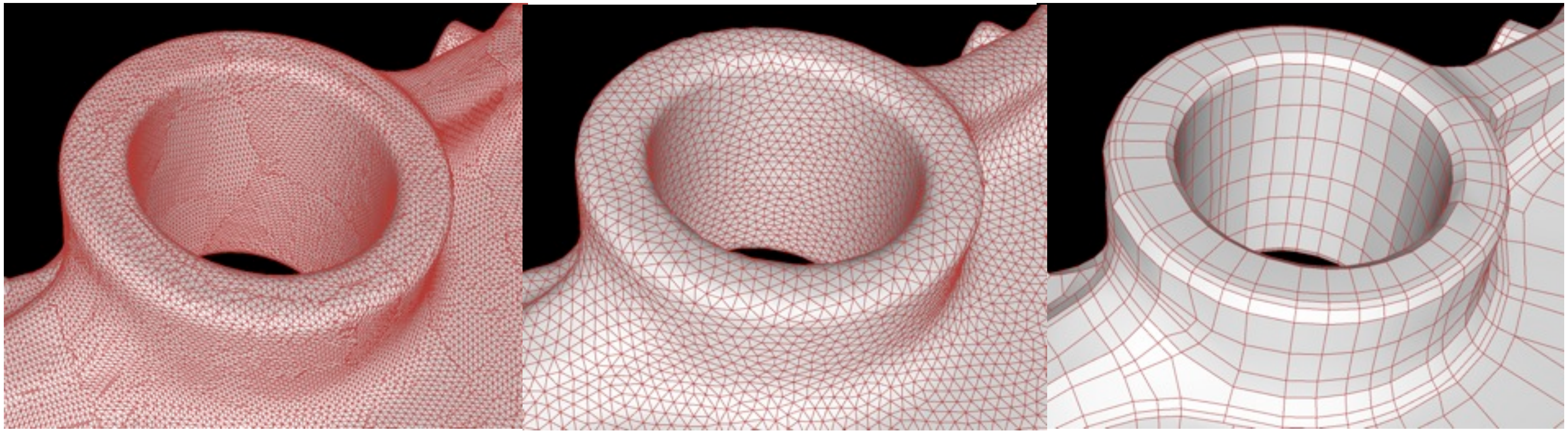
→ Remeshing
Pierre



Modifications

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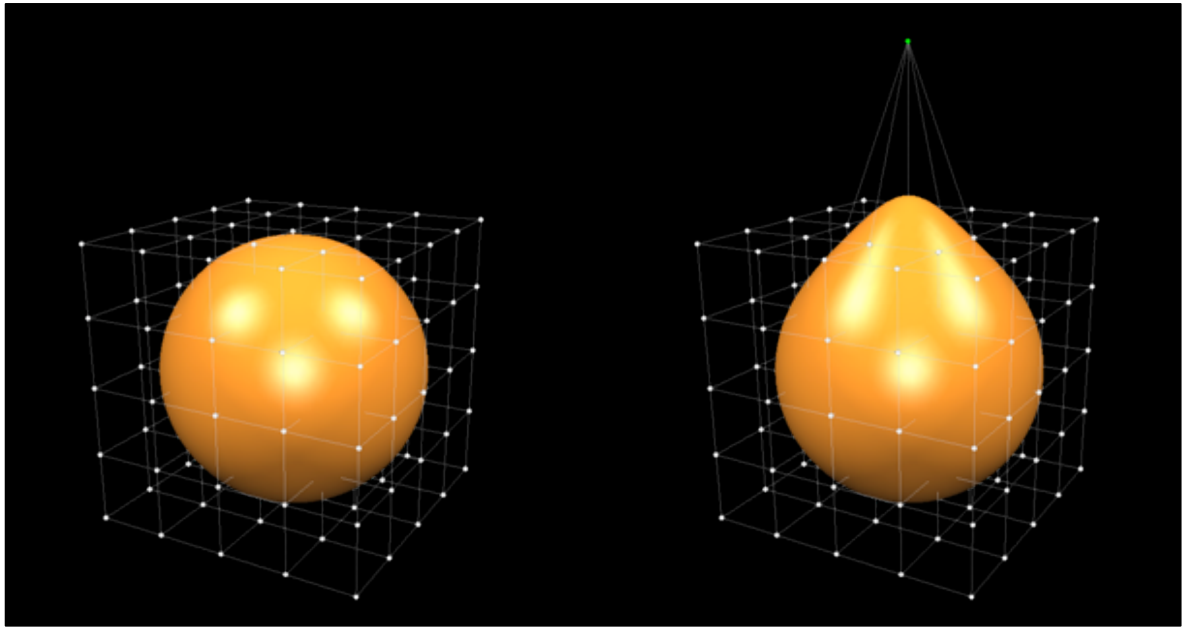

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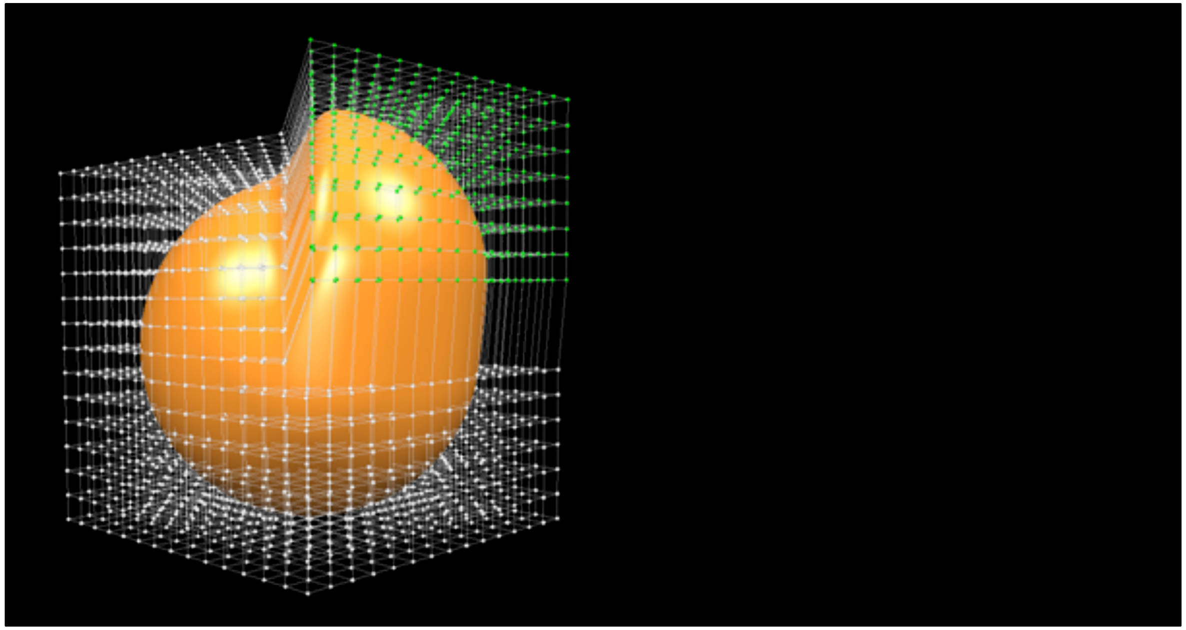

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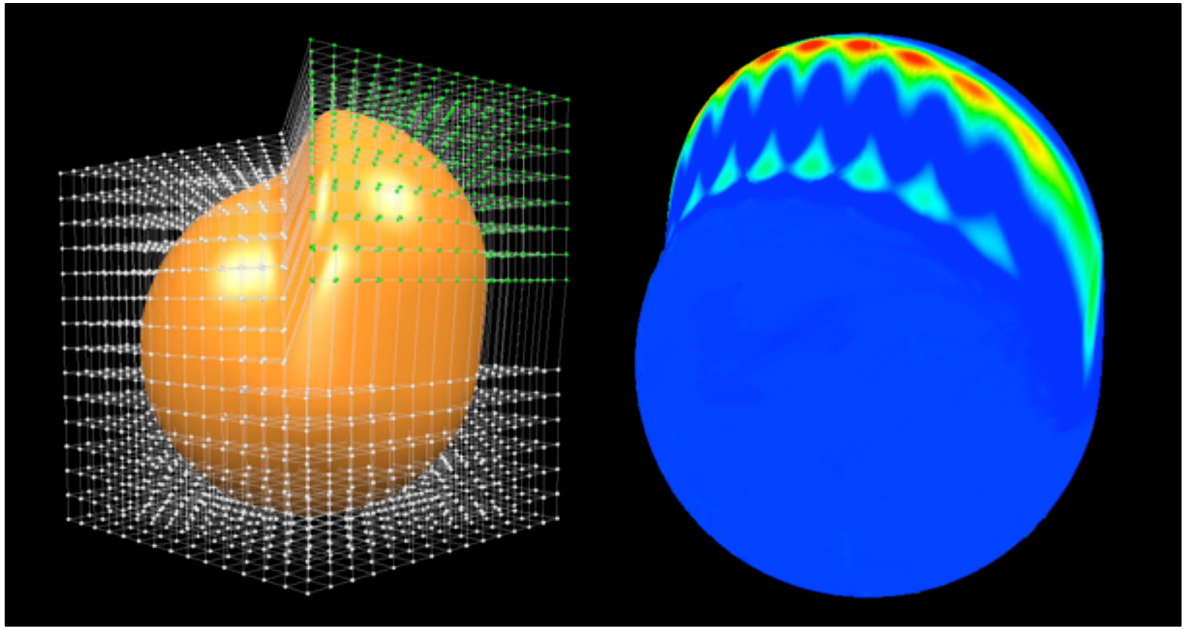

Modifications

- parameteric
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→ Mesh Editing
Mario